URSI GASS 2020, Rome, Italy, 29 August - 5 September 2020

On the use of Quasi-Helmholtz Projectors for the High Frequency Preconditioning of the Electric Field Integral Equation

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Abstract

Despite its modelling efficiency, the boundary element method (BEM) suffers from several sources of ill conditioning that reduce its accuracy and impede its applicability. The conditioning of the BEM matrix grows unbounded in three different regimes: (i) when the discretization density is constant and the frequency decreases, (ii) when the frequency remains constant and the discretization density increases and (iii) when the frequency increases while the mesh parameter remains at a fixed ratio of the wavelength. While the simultaneous stabilization of regime (i) and (ii) has been obtained leveraging on Calderón preconditioning and on the quasi-Helmholtz projectors, the last regime remains problematic. However, the intrinsic ill-conditioning caused by the frequency increase should not be confused with the periodic ill-conditioning caused by spurious resonances, which is out of the scope of this study. In this paper, instead, we present a scalar and a vector electric field integral formulation that are stable in all three regimes, symmetric, and do not require the computation of the dense matrices on the barycentric refinement of the geometry.

1 Introduction

The electric field integral equation (EFIE) is one of the most established schemes for characterising scattering phenomena by perfect electrically conducting (PEC) bodies. The popularity of this formulation, discretized with the boundary element method (BEM), follows from the reduced number of unknowns to be solved for, since only the scatterer's boundaries are discretized, from its automatic enforcement of the radiation conditions, and from its resilience to numerical dispersion. Despite these numerous advantages, because the system matrices are dense, the EFIE must be combined with fast solvers, such as the fast multipole method [1], to obtain a solution in linear complexity. The linear complexity of the resolution requires, however, the system matrices to be well-conditioned.

The electric field integral operator (EFIO), like most electromagnetic integral operators, is ill-conditioned in several regimes: (i) when the simulation frequency decreases while the discretization density is fixed and (ii) when the discretization density increases at a fixed frequency. The low frequency regime (i) and dense discretization regime (ii) have been widely studied and have satisfactory remedies. However, an ill-conditioning that can not be cured with the same techniques occurs when (iii) the frequency increases while the ratio between the mesh parameter and the wavelength remains constant.

The low frequency breakdown of the EFIE – regime (i) – is typically stabilized by decomposing the formulation into its solenoidal and non-solenoidal parts, to treat each part independently and make their behaviour compatible, thus fixing the root cause of the ill-conditioning. This Helmholtz decomposition can be obtained via loop-star approaches [2], that do however worsen the dense-discretization behaviour – regime (ii) – of the equation or via quasi-Helmholtz projectors [3] that do not share this drawback. Other approaches rely on the computation of auxiliary variables but incur a non-negligible computational overhead. The dense discretization behaviour has successfully been tackled by leveraging on the Calderón identities that recognize that the EFIO can fix its own instability. The simultaneous stabilization of both regimes has been obtained more recently by combining a Calderón approach with the quasi-Helmholtz projectors [3]. This technique, however, requires the dense electromagnetic operators to be computed on the barycentric refinement of the mesh. A subsequent work leverages on Helmholtz operators to attain this dual stabilization without barometrically refined electromagnetic operators [4].

In the high frequency regime two distinct sources of ill-conditioning should not be confused; the unbounded conditioning number degradation with increasing frequency (and corresponding discretization) and the periodic ill-conditioning caused by the spurious internal resonances of the EFIO are distinct phenomena. These spurious resonances are typically handled via a combined field integral equation (CFIE), and will not be treated in this contribution. We will instead focus on the intrinsic ill-conditioning of regime (iii).

In this paper, we introduce two new symmetric electric field integral formulations, one vector and one scalar, that exhibit a stable conditioning in all three regimes of interest: (i) the low frequency regime, (ii) the dense discretization regime and, and (iii) the high frequency regime. The stabilization of the high and low frequency behaviour of the formulation is obtained via quasi-Helmholtz projectors and carefully built Helmholtz operators, while the dense discretization behaviour will be cured using a Calderón-like approach. The resulting formulation does not require any barycentric refinement of dense operators. Some early developments and partial results have been previously reported in [5] and will be developed further in the present contribution. The validity of the schemes will be demonstrated through a spherical harmonics analysis and numerical experiments.

2 Notation and Background

The EFIE relates the surface current density $\boldsymbol{j}$ induced by an impinging electric field $\boldsymbol{E}^{\mathrm{i}}$ on the boundary Γ of a PEC object residing in a background medium of conductivity ε and permeability μ as

$$\eta\left(\mathscr{T}\boldsymbol{j}\right)(\boldsymbol{r}) = -\hat{\boldsymbol{n}}(\boldsymbol{r})\times\boldsymbol{E}^{\mathrm{i}}(\boldsymbol{r}), \tag{1}$$

where

$$\left(\mathscr{T}\boldsymbol{j}\right)(\boldsymbol{r}) = -\mathrm{j}k\left(\mathscr{T}_{\mathrm{s}}\boldsymbol{j}\right)(\boldsymbol{r}) - \frac{1}{-\mathrm{j}k}\left(\mathscr{T}_{\mathrm{h}}\boldsymbol{j}\right)(\boldsymbol{r}), \tag{2}$$

$$\left(\mathscr{T}_{\mathrm{s}}\boldsymbol{j}\right)(\boldsymbol{r}) = \hat{\boldsymbol{n}}(\boldsymbol{r})\times\int_\Gamma \frac{\mathrm{e}^{-\mathrm{j}k|\boldsymbol{r}-\boldsymbol{r}'|}}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|}\boldsymbol{j}(\boldsymbol{r}')\mathrm{d}S', \tag{3}$$

$$\left(\mathscr{T}_{\mathrm{h}}\boldsymbol{j}\right)(\boldsymbol{r}) = \hat{\boldsymbol{n}}(\boldsymbol{r})\times\nabla\int_\Gamma \frac{\mathrm{e}^{-\mathrm{j}k|\boldsymbol{r}-\boldsymbol{r}'|}}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|}\nabla'\cdot\boldsymbol{j}(\boldsymbol{r}')\mathrm{d}S', \tag{4}$$

$\eta = \sqrt{\mu/\varepsilon}$, $k = 2\pi f\sqrt{\mu\varepsilon}$ and $\hat{\boldsymbol{n}}(\boldsymbol{r})$ is the normal of Γ. To solve the equation with numerical methods, the unknown current density is expanded as a linear combination of Rao-Wilton-Glisson (RWG) basis functions $\{\boldsymbol{f}_i\}$, $\boldsymbol{j}(\boldsymbol{r}) \approx \sum_{i=0}^{N}[\boldsymbol{j}]_i\boldsymbol{f}_i(\boldsymbol{r})$ and the resulting equation is tested with rotated RWG functions $\{\hat{\boldsymbol{n}}\times\boldsymbol{f}_i\}$. Finally, the linear system is

$$\eta T\boldsymbol{j} = \boldsymbol{e}^{\mathrm{i}}, \tag{5}$$

where $T = -\mathrm{j}kT_s - (-\mathrm{j}k)^{-1}T_h$, $[T_{\mathrm{s}}]_{ij} = \langle\hat{\boldsymbol{n}}\times\boldsymbol{f}_i, \mathscr{T}_{\mathrm{s}}\boldsymbol{f}_j\rangle$, $[T_{\mathrm{h}}]_{ij} = \langle\hat{\boldsymbol{n}}\times\boldsymbol{f}_i, \mathscr{T}_{\mathrm{h}}\boldsymbol{f}_j\rangle$, and $\left[\boldsymbol{e}^{\mathrm{i}}\right]_i = \langle\hat{\boldsymbol{n}}\times\boldsymbol{f}_i, -\hat{\boldsymbol{n}}\times\boldsymbol{E}^{\mathrm{i}}\rangle$. The following section will involve several families of basis function and their dual: the RWG basis functions $\boldsymbol{f}$ and the Buffa–Christiansen functions $\widetilde{\boldsymbol{f}}$ [6], the patch basis functions p and their dual $\widetilde{p}$, and the pyramid basis functions λ and their dual $\widetilde{\lambda}$. Along with these basis functions we define the corresponding Gram and mixed-Gram matrices as

$$[G_{ab}]_{ij} = \langle a_i, b_j\rangle, \tag{6}$$

where a and b represent potentially different families of basis functions.

3 New formulations

To overcome the three breakdown of the EFIE we introduce a new formulation, vector in nature, intended as a drop-in replacement of the standard equation, and a second one designed to replace an EFIE that has already undergone a loop-star decomposition.

In the vector formulation, the Helmholtz decompositions are obtained through the quasi-Helmholtz projectors $P_\Lambda = \Lambda(\Lambda^{\mathrm{T}}\Lambda)\Lambda^{\mathrm{T}}$, which projects to the solenoidal subspace and $P_\Sigma = \Sigma(\Sigma^{\mathrm{T}}\Sigma)\Sigma^{\mathrm{T}}$ which projects to the non-solenoidal subspace. In particular, the projectors are used to obtain the Helmholtz decomposition of the $\hat{\boldsymbol{n}}\times\mathscr{T}$ operator. The solenoidal and non-solenoidal parts of the operator are then carefully preconditioned using vector Helmholtz operators to render them immune to problems (i), (ii), and (iii). For the non-solenoidal operator we obtain

$$k^2\hat{\boldsymbol{n}}\times\mathscr{T}\left(\boldsymbol{\Delta}+k_{\mathrm{m}}^2\mathscr{I}\right)^{-1}\hat{\boldsymbol{n}}\times\mathscr{T}, \tag{7}$$

while for its solenoidal counterpart we have

$$k^{-2}\hat{\boldsymbol{n}}\times\mathscr{T}\left(\boldsymbol{\Delta}+k_{\mathrm{m}}^2\mathscr{I}\right)\hat{\boldsymbol{n}}\times\mathscr{T}, \tag{8}$$

in which $\boldsymbol{\Delta}$ is the vector Laplacian defined on Γ and $k_{\mathrm{m}} = k + 0.4\mathrm{j}k^{1/3}R^{-2/3}$ is a wavenumber that makes it possible to precondition the EFIO with a modified vector Helmholtz operator on a radius R sphere [7]. The resulting operator is discretized, following a Galerkin approach, as

$$Z^v = T_{\Lambda\Sigma}^{\mathrm{T}}\left(G^{-1}_{\widetilde{\boldsymbol{f}},\hat{\boldsymbol{n}}\times\boldsymbol{f}}H_\Lambda G^{-1}_{\hat{\boldsymbol{n}}\times\boldsymbol{f},\widetilde{\boldsymbol{f}}} + H_\Sigma^+\right)T_{\Lambda\Sigma}, \tag{9}$$

where, for readability, we have defined

$$\widetilde{L}_\Lambda = G^{-1}_{\lambda\widetilde{p}}G_{\lambda\lambda}\left(\Lambda^{\mathrm{T}}G_{\boldsymbol{f}\boldsymbol{f}}\Lambda\right)^+G_{\lambda\lambda}G^{-1}_{\widetilde{p}\lambda}, \tag{10}$$

$$\widetilde{L}_\Sigma = G^{-1}_{\widetilde{\lambda}p}G_{\widetilde{\lambda}\widetilde{\lambda}}\left(\Sigma^{\mathrm{T}}G_{\widetilde{\boldsymbol{f}}\widetilde{\boldsymbol{f}}}\Sigma\right)^+G_{\widetilde{\lambda}\widetilde{\lambda}}G^{-1}_{p\widetilde{\lambda}}, \tag{11}$$

$$H_\Lambda = \Lambda\left(-G^{-1}_{\widetilde{p}\widetilde{p}} + k_{\mathrm{m}}^2\widetilde{L}_\Lambda\right)\Lambda^{\mathrm{T}}, \tag{12}$$

$$H_\Sigma = \Sigma\left(-G^{-1}_{pp} + k_{\mathrm{m}}^2\widetilde{L}_\Sigma\right)\Sigma^{\mathrm{T}}, \tag{13}$$

$$T_{\Lambda\Sigma} = \left((-\mathrm{j}k)^{-1}P_\Lambda + P_\Sigma\right)T\left(P_\Lambda - \mathrm{j}kP_\Sigma\right). \tag{14}$$

For this discretization to be stable at arbitrarily low frequencies, the terms $T_{\mathrm{h}}P_\Lambda$, $P_\Lambda T_{\mathrm{h}}$, $H_\Sigma^+P_\Lambda$, $P_\Lambda H_\Sigma^+$, $H_\Lambda G^{-1}_{\hat{\boldsymbol{n}}\times\boldsymbol{f},\widetilde{\boldsymbol{f}}}P_\Sigma$ and $P_\Sigma G^{-1}_{\widetilde{\boldsymbol{f}},\hat{\boldsymbol{n}}\times\boldsymbol{f}}H_\Lambda$ must be explicitly set to $\boldsymbol{0}$.

The scalar formulation can be derived in a similar fashion: the solenoidal and non-solenoidal parts of $\hat{\boldsymbol{n}}\times\mathscr{T}$

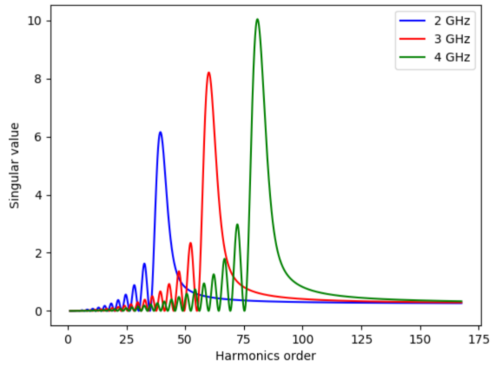
$$\mathscr{T}_\Lambda = \nabla\cdot\hat{\boldsymbol{n}}\times\left(\hat{\boldsymbol{n}}\times\mathscr{T}\right)\hat{\boldsymbol{n}}\times\nabla, \tag{15}$$

$$\mathscr{T}_\Sigma = \Delta^{-1}\nabla\cdot\left(\hat{\boldsymbol{n}}\times\mathscr{T}\right)\nabla\Delta^{-1}, \tag{16}$$

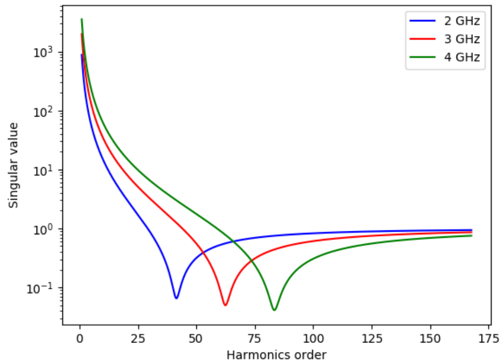
are stabilized in regimes (i) to (iii) by multiplying them with Helmholtz operators of opposite behaviour, which yields the loop-star blocks

$$k^{-2}\mathscr{T}_\Lambda\Delta^{-1}\left(\Delta+k_{\mathrm{m}}^2\mathscr{I}\right)\Delta^{-1}\mathscr{T}_\Lambda, \tag{17}$$

$$k^2\mathscr{T}_\Sigma\Delta\left(\Delta+k_{\mathrm{m}}^2\mathscr{I}\right)^{-1}\Delta\mathscr{T}_\Sigma. \tag{18}$$



(a)



(b)

Figure 1. Spectrum of (a) $k^{-2}\mathcal{T}_\Lambda\Delta^{-1}\mathcal{T}_\Lambda$ at different frequencies showing an unbounded growth of the highest singular value and (b) $\left(\Delta+k_\mathrm{m}^2\mathcal{I}\right)\Delta^{-1}$ showing an opposite behaviour.

The key concept behind this preconditioning is to build surrogate operators from traditional Helmholtz operators, that exhibit a spectral behaviour that is the opposite of the problematic spectrum of the electromagnetic operator. For instance while $k^{-2}\mathcal{T}_\Lambda\Delta^{-1}\mathcal{T}_\Lambda$ is of pseudo-differential order 0 and low frequency stable, it still has an high frequency ill-conditioning (Figure 1a). By multiplying it by $\left(\Delta+k_\mathrm{m}^2\mathcal{I}\right)\Delta^{-1}$ that has an opposite behaviour (Figure 1b), a well-conditioned operator can be obtained. The resulting scalar equation is discretized in block form as

$$Z^s = T_\mathrm{LS}^\mathrm{T}\begin{bmatrix} L_\mathrm{L}^+ H_\mathrm{L} L_\mathrm{L}^+ & \mathbf{0} \\ \mathbf{0} & G_{\widetilde{\lambda}p}^{-1} L_\mathrm{S} H_\mathrm{S}^+ L_\mathrm{S} G_{p\widetilde{\lambda}}^{-1}\end{bmatrix} T_\mathrm{LS}\,, \tag{19}$$

where the different blocks are built out of the sub-matrices

$$L_\mathrm{L} = -\Lambda^\mathrm{T} G_{ff}\Lambda\,, \tag{20}$$

$$L_\mathrm{S} = -\Sigma^\mathrm{T} G_{\widetilde{f}\widetilde{f}}\Sigma\,, \tag{21}$$

$$H_\mathrm{L} = L_\mathrm{L} + k_\mathrm{m}^2 G_{\lambda\lambda}\,, \tag{22}$$

$$H_\mathrm{S} = L_\mathrm{S} + k_\mathrm{m}^2 G_{\widetilde{\lambda}\widetilde{\lambda}}\,, \tag{23}$$

$$\widetilde{\Sigma} = \Sigma\left(\Sigma^\mathrm{T}\Sigma\right)^+ G_{pp}\,, \tag{24}$$

$$T_\mathrm{LS} = \begin{bmatrix}(-\mathrm{j}k)^{-1}\Lambda^\mathrm{T} \\ \widetilde{\Sigma}^\mathrm{T}\end{bmatrix} T\begin{bmatrix}\Lambda & -\mathrm{j}k\widetilde{\Sigma}\end{bmatrix}. \tag{25}$$

In this formulation too, the low frequency stability of the discretization requires that the terms $\Lambda^\mathrm{T}T_\mathrm{h}$ and $T_\mathrm{h}\Lambda$ be explicitly set to $\mathbf{0}$. The new scalar equation (19), while stable, has a nullspace of dimension 2 spanned by the constant vector that can be removed by proper deflections, however these derivations are left out for brevity.

4 Numerical Results

To verify their validity we have performed a spherical harmonics analysis of the two formulations. On a sphere of radius R, the new operators (17) and (8) can be shown to admit as eigenvectors Y_{lm} and $\hat{\boldsymbol{n}}\times\nabla Y_{lm}$, respectively, where Y_{lm} is the spherical harmonic of order l with m taking values in $[-l, l]$. While different, these eigenvectors are associated to the same eigenvalues

$$\sigma_\Lambda(l,k) = k^{-2}\left(J_l(kR)H_l^{(2)}(kR)\right)^2\left(-\frac{l(l+1)}{R^2}+k_\mathrm{m}^2\right), \tag{26}$$

where $H_l^{(2)}$ is the Riccati-Hankel function and J_l is the Riccati-Bessel function. A similar analysis of the non-solenoidal operators (18) and (7) provides the eigenvalues

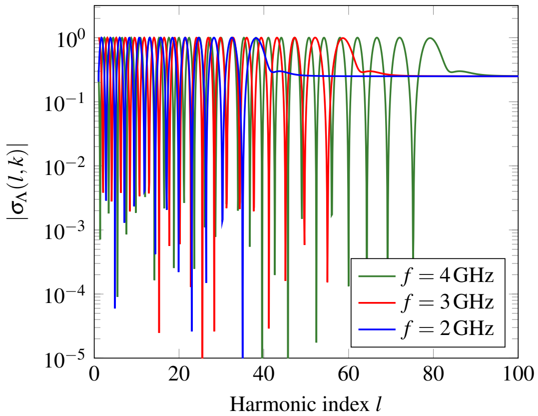
$$\sigma_\Sigma(l,k) = k^2\left(J_l{}'(kR)H_l^{(2)}{}'(kR)\right)^2\left(-\frac{l(l+1)}{R^2}+k_\mathrm{m}^2\right)^{-1}. \tag{27}$$

The eigenvalues obtained though this analysis (Figures 2a and 2b) demonstrate the stability of the operators in the high frequency regime since their maximum eigenvalue remains bellow one, while the convergence of the elliptic to $1/4$ indicates their dense discretization stability.

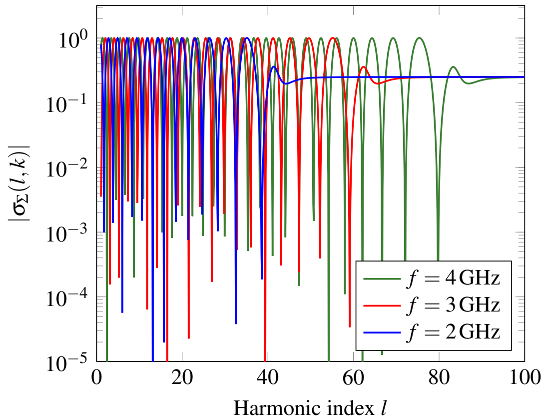
To further confirm the stability of our schemes beyond spheres, we have studied the evolution of the condition number of the vector formulation on a complex geometry, in the dense discretization (Figure 3a) and low frequency (Figure 3b) regimes. In both cases the new formulation behaves as expected and remains well-conditioned, while the standard EFIE becomes severely ill-conditioned.

5 Conclusion

We have introduced two new electric type integral equations that overcome the limitations of the standard EFIE in the low frequency, dense discretization, and high frequency regimes. These stabilized formulation, one vector and one scalar, are built by preconditioning the original EFIO (or its solenoidal and non-solenoidal components) with Helmholtz operators that are computed with a modified wave number. The vector formulation serves as a replacement of the standard EFIE, while the second one is intended to replace a loop-star decomposed EFIE. These formulations have the advantage of only requiring sparse operators to be computed on the barycentric mesh and not the dense electromagnetic ones.

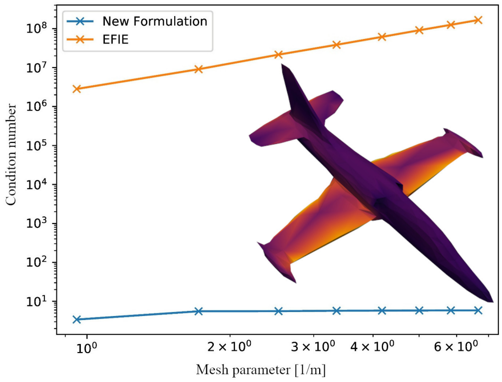


(a)

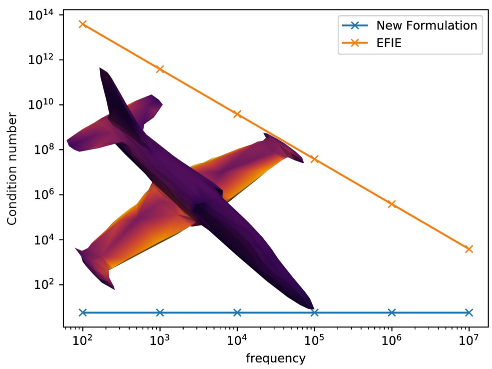


(b)

Figure 2. Absolute value of the (a) solenoidal and (b) non-solenoidal eigenvalues of the new vector and scalar formulations at different frequencies.



(a)



(b)

Figure 3. Comparison of the conditioning of the new vector formulation against the standard EFIE on the structure illustrated as insert in the dense discretization (a) and low frequency (b) regimes.

6 Acknowledgements

This work was supported by the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No 724846, project 321).

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