On a Method of Solution to Dispersion Equations in Electromagnetics

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Abstract

A possible general approach to the analysis of dispersion equations (DEs) of electromagnetics is presented. The method takes into account the known explicit forms of DEs describing eigenoscillations and normal waves in layered structures and is based on the development of the notion of generalized cylindrical polynomials. The approach enables one to complete rigoroius proofs of existence and determine domains of localization of the DE roots and validate iterative numerical solution techniques.

1 Introduction

Determination of real and complex waves is reduced to multi-parameter eigenvalue problems [1]–[3] and then in many cases (when e.g. the Helmholtz equation admits closed-form solutions using separation of variables) to dispersion equations (DEs). Their complete study is a sophisticated task which requires deep analytical-numerical investigations. In this work, a major attention is paid to creating an introduction to a general method that enables obtaining (sufficient) conditions of the existence and description of localization of the DE roots providing thus justification for the methods of determination of oscillations and waves in multi-layered structures which can be easily implemented in calculations. The method is based on the studies of the so-called *generalized cylindrical polynomials (GCPs)* aimed in particular to finding zeros of GCPs using different approaches.

2 Dispersion Equations

For waveguides possessing circular symmetry, like a dielectric waveguide (DW) or a Goubau line (GL) formed by several concentric layers of media, all the field components of symmetric and nonsymmetric wave can be expressed [2, 3] via potential function $\phi_m(r)$ which is generally a linear combination of several cylindrical functions of order m. This gives rise to explicit forms of the DEs for (single-layer) DW and GL [3]-[6]

$$F_d(x) \equiv P_D(w)J_1(x) + xJ_0(x) = 0 \quad \text{(DW)}, \tag{1}$$
$$F_g(x) \equiv P_D(qw)\Phi_1(x) - qx\Phi_0(x) = 0 \quad \text{(GL)}, \tag{2}$$

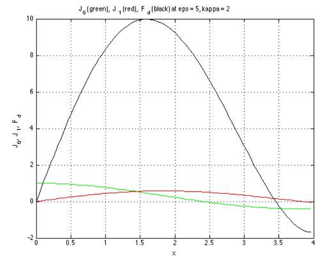

Figure 1. Example for a GCP $F_d(x)$ in DE (1): plots of $J_0(x)$ (green), $J_1(x)$ (red), and $F_d(x)$ (black) at $\varepsilon = 5$ and $\kappa = 2$ ($u = \kappa\sqrt{\varepsilon - 1} = 4$) displaying a zero of $F_d(x)$ between neighboring zeros of $J_1(x)$ and neighboring zeros of $J_0(x)$ and $J_1(x)$.

where J_m, Y_m and K_m ($m = 0, 1$) are the Bessel, Neumann, and Macdonald functions, $P_D(w) = \varepsilon\dfrac{wK_0(w)}{K_1(w)}$, $\Phi_0 = J_0(qx)Y_0(x) - J_0(x)Y_0(qx)$, $\Phi_1 = J_0(x)Y_1(qx) - J_1(qx)Y_0(x)$, and the waveguide geometric and material parameters $x = \kappa\sqrt{\varepsilon - \gamma^2}$, $\gamma = \frac{\beta}{k_0}$, $\kappa = k_0 a$, $u = \kappa\sqrt{\varepsilon - 1}$ (β, ε, and k_0 are, respectively, the wave progation constant, permittivity, and free-space wavenumber), $w = \sqrt{u^2 - x^2} = \kappa\sqrt{\gamma^2 - 1}$, and $q = \frac{b}{a} > 1$ (a and b are characteristic dimensions of DW and GL). Straightforward analysis of (1) and (2) demonstrates that functions F_d and F_g entering DEs have distinct common features: they are sums of (products of) cylindrical functions J_m and Φ_m each having infinitely many alternating simple positive zeros. The latter yields an immediate proof (illustrated by Fig. 1 and verified below) of the (sufficient conditions) provoding existence of real roots of the DEs located between zeros of J_m and Φ_m ($m = 0, 1$). The existence, localization, and number of the DE roots are governed actually by a number of zeros of J_m or Φ_m that are inside the domain $x \in (0, u)$ of F_d and F_g; that is, by the value of parameter u.

3 General Approach

The facts reported above as well as explicit expressions for the DEs obtained in [2] and [4, 5] for multi-layered DWs and GLs (open or shielded) suggest that DEs can be represented in the general form of weighted linear combinations

$$\mathscr{G}_{N,M}(x) = \sum_{m=1}^{M} P^{(m)}(k^{(m)}x)\mathscr{W}_N^{(m)}(x) \tag{3}$$

of the products $\mathscr{W}_N^{(m)} = \Pi_{n=0}^N \mathscr{I}_n^{(m)}(\kappa_n^{(m)} x)$ of cylindrical functions $\mathscr{I}_n^{(m)}$, where $P_N^{(m)}(x)$ are constants or bounded continuous functions for $x > 0$ (determined explicitly in DEs for multi-layered waveguides with circular [4, 5] or planar [7, 8] symmetry), and $\kappa_n^{(m)}$ and $k^{(m)}$ are real parameters (quantities depending on parameters of the structure in a DE). Such functions are referred to as GCPs of order (M,N).

In order to formulate sufficient conditions that guarantee the existence of zeros of a GCP and describe their localization, we will use the following

Statement. Let $f_j(x) \in C[a,b]$, $j = 1,2,3$ (continuous in a closed interval $[a,b]$), $f_2(a) = f_2(b) = 0$, and $f_1(a)f_1(b) < 0$ or, equivalently, there is one $s \in (a,b)$ such that $f_1(s) = 0$; then the equation

$$f(x) \equiv f_2(x)f_3(x) + f_1(x) = 0$$

has a root $x = x^* \in (a,b)$.

This statement can be applied to validate a recursive procedure of proving the existence and determining zeros of GCPs.

To this end, note that each $\mathscr{W}_N^{(m)}(x)$ has (as a function of x) infinitely many positive zeros forming a countable set $Z_N^{(m)} = \cup_{n=0}^N Z_n^{(m)}$ being a union of the sets $Z_n^{(m)}$ of zeros $z_{n,k}^{(m)}$ $(k = 1,2,\dots)$ of all $\mathscr{I}_n^{(m)}$. Elements of $Z_n^{(m)}$ (and $Z_N^{(m)}$) depend on the parameter vector $\bar{\kappa}_N = (\kappa_1^{(m)},\dots,\kappa_N^{(m)})$. Next, represent GCP (3) as

$$\mathscr{G}_{N,M}(x) = P^{(M)}(k^{(M)}x)\mathscr{W}_N^{(M)}(x) + \mathscr{G}_{N,M-1}(x), \qquad (4)$$

where $\mathscr{W}_N^{(M)}(x)$ vanishes at the endpoints of the interval $I_{n,k}(m) = (a,b) = (z_{n,k}^{(m)}, z_{n,k+1}^{(m)})$ between every two its neighboring zeros. Assuming that the parameter vector $\bar{\kappa}_N$ is such that GCP $\mathscr{G}_{N,M-1}(x)$ of order $(M-1,N)$ has a zero on $I_{n,k}^{(m)}$, we can use Statement to conclude that GCP (3) of order (M,N) also has a zero on this interval.

For a GCP F_g in (2) we can apply this reasoning and Statement by setting $f_1(x) = -qx\Phi_0(x)$ and $f_2(x) = \Phi_1(x)$. Then $F_g(x)$ has a zero between every two neighboring zeros $h_k^{(1)}$ and $h_{k+1}^{(1)}$ of $\Phi_1(x)$ as soon as $q > 1$ is such that a zero of $\Phi_0(x)$ belongs to the interval $(h_k^{(1)} h_{k+1}^{(1)})$. The latter condition can be satisfied because zeros $h_k^{(j)} = h_k^{(j)}(q)$ of $\Phi_j(x)$ alternate for different $j = 1,2$ and form sequences of points decreasing with respect to q [3, 4]. The conclusion concerning the existence and location of the zeros of F_g between neighboring alternating zeros of $\Phi_0(x)$ and $\Phi_1(x)$ is perfectly illustrated by Fig. 2.

Existence and analysis of complex oscillations and waves in terms of the solution to DEs in the complex domain can be performed using the general approach and results outlined in [7] and [8].

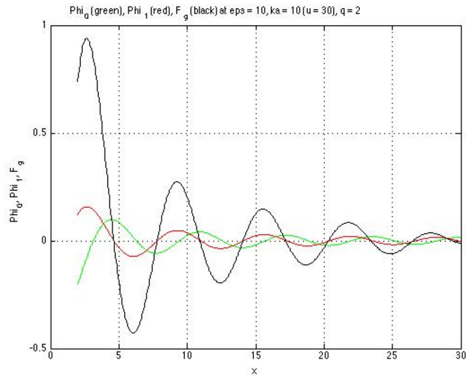

Figure 2. Zeros of a GCP F_g (black) given by (2) situated between neihboring zeros of Φ_0 (green) and Φ_1 (red).

4 Conclusion

We have proposed a method for verifying the existence and determination of location of roots of the DEs expressed as weighted sums of products of cylindrical functions. The results complete mathematical theory of DEs for multi-layered waveguides possessing circular symmetry and can be extended to more general structures as well as to determination of complex waves .

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