Stochastic Finite Element Method for Electromagnetic Material Property Variations Over Multiple Subdomains

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Abstract

The electromagnetic system response distribution due to the stochastic variation in properties of media such as relative permittivity and loss tangent, for multiple discrete subdomains of the problem domain, has been estimated by spectral representation of random system parameters . The finite element formulation for electromagnetics with edge elements are updated to extend the model to include these stochastic variations. The comparison of numerical results with Monte Carlo simulation done with large number of samples implies that the proposed method is computationally efficient and extremely fast.

1 Introduction

Variability in modelling parameters have always been a challenge in the realization of microwave systems due to fabrication tolerances and environmental uncertainties. The material properties of the dielectrics in circuits are not generally a fixed value as assumed by the mathematical EM models. There will be a stochastic variation along the material body due to the inconsistencies in the fabrication process. On the other hand, the operating environment play an important role in the working of high frequency circuits. The thermal coefficient of dielectric constant quantifies the impact of temperature on the permittivitty of a material, which gives an estimate on the deviation of expected behaviour of a microwave circuit with changes in temperature. [1] Due to the environmental conditions, military systems such as ballistic missiles, endure frequent temperature variations. Passive components, such as filters suffer from unwanted passbands and stopbands shifts with changes in permittivity, while active components may be pushed to unstable regions [2]. The aim of this paper is not to estimate these tolerance limits, but the effects of the stochasticity introduced in the system parameters due to these tolerances.

The aforementioned deviations are not generally addressed by EM simulation tools due to the impractical run time taken by the commonly used Mote Carlo methods. Such probabilistically varying EM systems can be efficiently solved in frequency domain by spectral stochastic finite element method. Analysis of a finite element based system with the spectral decomposition of probabilistically varying parameters as a random field is called spectral stochastic finite element method [3]. This is more efficient compared to other methods in terms of computational time complexity. Stochastic FEM is extensively used in civil and mechanical engineering for reliability and risk analysis. However, stochastic FEM methods for EM problems are less explored. Material variation is analysed using spectral expansion in FDTD is designed by Edwards et al. [4]. Stochastic magnetostatics and electrostatic problem are solved using SSFEM by R. Gaignaire et al. [5]. Magnetostatic problems with A vector potential formulation involving parameter uncertainties are investigated by K. Beddek et al. [6].

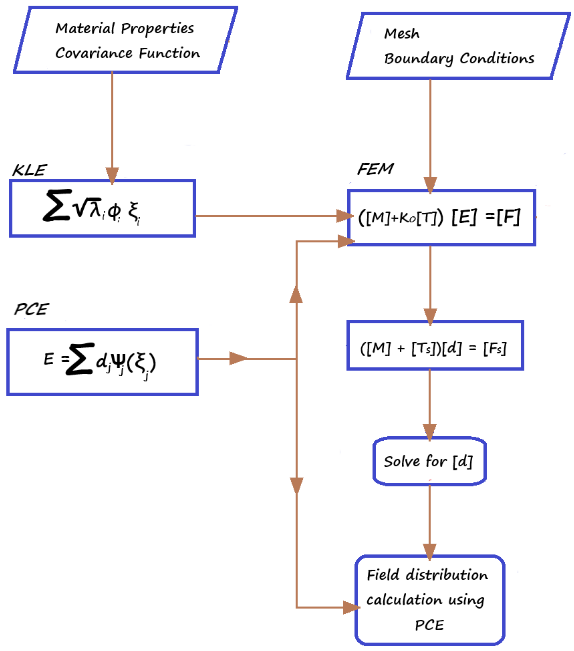

Figure 1. SSFEM Flow Diagram

This paper proposes an edge element based SSFEM for material uncertainties in multiple subdomains in a general frequency domain 3D fullwave EM scattering problem. Numerical analysis has been done by varying complex permittivity of media with different standard deviations and mean, at two separated regions of the problem domain. The SSFEM results are compared to the Monte Carlo results for validation.

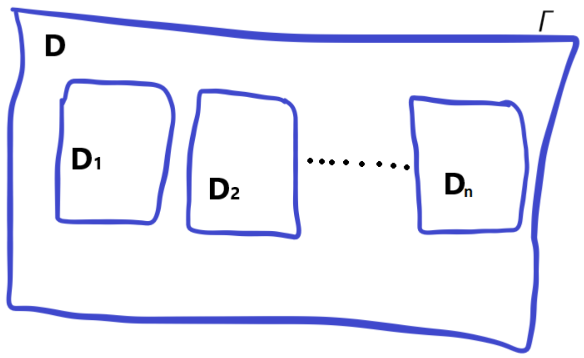

Figure 2. Problem Domain: D is the complete physical domain and $D_1, D_2 \ldots D_n$ are subdomains where the stochastic variations are assumed

2 Formulation

A boundary value problem in the full-wave scattering analysis with inhomogeneous media is modelled by [7],

$$\nabla \times \big(\frac{1}{\mu_r}\nabla \times E\big) - k_0^2 \varepsilon_r E = 0 \quad in\, \mathbf{D} \tag{1}$$

Where $\mathbf{D}$ is the computational domain with boundary Γ as shown in Fig. (2). Γ can be a Dirichlet , Neumann or Robin boundary.

Using tetrahedral edge elements the system equations in eq. (1) can be reduced to a linear system $[K][E] = [b]$. Here $[K]$ is obtained by assembling the individual elemental matrices $[K^e]$ over the entire domain. $[K^e]$ and $[b^s]$ matrix entries can be derived as,

$$[K^e] = [M^e] + k_0^2 \varepsilon_r [T^e] \tag{2}$$

$$[b^s] = \iint_{S^s} S^s.(U^{inc} \times \hat{n})ds \tag{3}$$

where,

$$[M^e] = \iiint_{V^e} \frac{1}{\mu_r^e} (\nabla \times N_i^e).(\nabla \times N_j^e)dv \tag{4}$$

$$[T^e] = -\iiint_{V^e} (N_i^e).(N_j^e)dv \tag{5}$$

The material properties of a subdomains D_1, D_2 ... D_n are independently random. Any random field over a subdomain D_i can be spectrally expanded using a linear combination of deterministic spatial functions of standard random variables [3].

The uncertainty in material in SSFEM is incorporated by taking ε_r in (1) as a random field over, either a part of or the complete problem domain with a known covariance function [8].

$$\varepsilon_r(x,\theta) = \mu(x) + \sum_{i=1}^{\infty} \sqrt{\lambda_i}\Phi_i(x)\xi_i(\theta) \quad x \in \mathbf{D},\ \theta \in \Omega \tag{6}$$

where $\{\xi_i\}$ represents the coordinates of realization of the random field with respect to the eigenpairs of the covariance function, $\{\lambda_i,\ \phi_i(x)\}$ over domain D. It is possible to truncate the series expansion in eq. (6) to include only a finite number of significant terms [8]. When there are multiple regions with different media properties with independent random variations, the KLE has to be evaluated for each of these domains with its covariance function. This has been represented with the following equation,

$$\varepsilon_{rk}(x,\theta) = \mu(x) + \sum_{i=1}^{\infty} \sqrt{\lambda_i}\Phi_{ki}(x)\xi_{ki}(\theta) \quad x \in \mathbf{D_k},\ \theta \in \Omega \tag{7}$$

ε_{rk} is the discretisation of random media property of domain D_k for $k = 1,2\ldots n$, $\{\xi_k\}$ and $\{\lambda_k,\ \phi_k(x)\}$ are defined for the respective regions.

The electric field stochastic variation is unknown, hence it is represented as a polynomial chaos expansion.

$$[E^e]^s = \sum_{j=1}^{P} d_j \Psi_j(\xi_r) \tag{8}$$

where P is the number of basis polynomial, $d_j(x)$ is the vector modal coefficients, $\{\xi_j\}$ is the set of standard orthogonal random variables and $\{\Psi_j(\xi_j)\}$ is a set of orthogonal polynomials.

The KL expansion has to be substituted instead of the permittivity of the eqn. (1). This has to done carefully such that since the KL expansion exixts only for the random domains and for each random domain the expansion is different due to the difference in material distributions. After proper substitution of equations (6 - 8) in edge FEM formulation and using the orthogonal properties of polynomial chaos basis by applying Galerkin's approximation [9], by which the residue is orthogonally projected to the square integrable random space, we get the matrix form,

$$\begin{bmatrix} [K^e]+[T^e_{s11}] & [T^e_{s12}] & \cdots & [T^e_{s1P}] \\ [T^e_{s21}] & [K^e]+[T^e_{s22}] & \cdots & [T^e_{s2P}] \\ \vdots & \vdots & \ddots & \vdots \\ [T^e_{sP1}] & [T^e_{sP2}] & \cdots & [K^e]+[T^e_{sPP}] \end{bmatrix} \begin{bmatrix} d_1 \\ \vdots \\ d_P \end{bmatrix} = \begin{bmatrix} F \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{9}$$

Where,

$$[T^e_{sij}] = \begin{cases} \left(\sum_{i=1}^{N_{KL}} \sqrt{\lambda_{ki}}\Phi^e_{ki}\xi_{ki}(\theta)\right) k_0^2 [T^e], & \text{if } if\ T \in D_k \\ 0, & \text{otherwise} \end{cases} \tag{10}$$

On assembling the above elemental formulation over the domain, the matrix system is obtained as,

$$\big([K_s] + [T_s]\big)[d] = [F_s] \tag{11}$$

Where $[K_s]$ is an $NP \times NP$ block diagonal matrix with $[K]_{N \times N}$ as the diagonal block and $[T_s]$ is an $NP \times NP$ matrix obtained by assembling $[T^e_{sij}]$. Each of the P blocks of

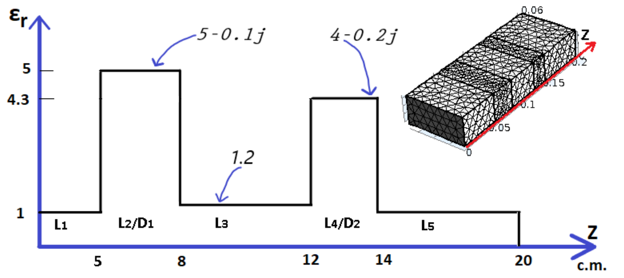

Figure 3. Waveguide dielectric fill profile along propagation direction

$N \times 1$ elements in $[d]$ is mapped to the edges in the physical model. Equations (9 - 11) model physical as well as the stochastic implications of the EM problem. After solving the system the stochastic distribution of electric field can be evaluated using PCE.

In the proposed approach the randomly varying properties using KLE and the system unknowns represented as a PCE are introduced into the FEM equations. The solution for the system is the spectral components of the stochastic response, which is used to reconstruct the probabilistic variations of the system solution as shown in (1).

In order to generate the eigen system for KLE, the finite element mesh can be reused. The typical element size in the random field, mesh size is close to half of the correlation length of the covariance function [8]. The KLE has to be evaluated for each of the stochastically varying region D_i (dielectric slabs in the above example) of the problem domain. The KLE can be evaluated using a coarser mesh and interpolated to the required points to increase computational efficiency. The evaluation of $[K_s]$ and $[T_s]$ has been done independently. Each $N \times N$ blocks in stochastic formulation is calculated by multiplying them with c_{ijk}. This leads to an efficient assembling process of SSFEM matrix. The sparsity of the assembled matrix helps in storing the matrix efficiently and achieving fast solution. The block structure of the matrix also enables solution using fast iterative procedures [10].

3 Numerical Results

A waveguide with 5 dielectric layers is taken as an example here. The dielectric constant varies along the propagation direction as shown in Fig 3. This is similar to a sandwich radome inside a waveguide. The dielectric constant is kept constant in layers L_1, L_3 and L_5 while the permittivity of the media is varied stochastically in layers L_2 and L_4, henceforth denoted by D_1 and D_2 respectively. The dielectric constant is stochastically varied in domains D_1 and D_2 independently with different covariance functions. For D_1 the complex relative permittivity is varied with a standard deviation of $0.2 + 0.02j$ and in D_2 a the complex relative permittivity is varied with a covariance of $0.16 + 0.045j$. This brings around a maximum 25% deviation in the dielectric constant and 100% deviation in loss tangent.

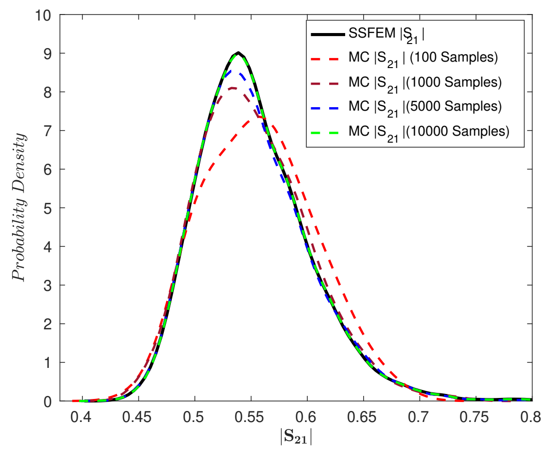

Figure 4. Probability distribution of transmission coefficient($|S_{21}|$). Comparison of SSFEM and Monte Carlo method with sample sizes 100,1000, 5000 and 10000.

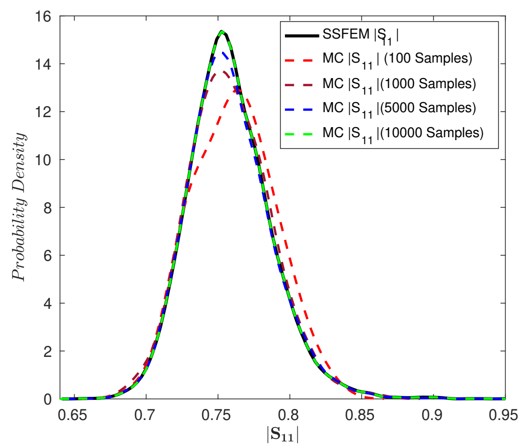

Figure 5. Probability distribution of reflection coefficient($|S_{11}|$). Comparison of SSFEM and Monte Carlo method with sample sizes 100,1000, 5000 and 10000.

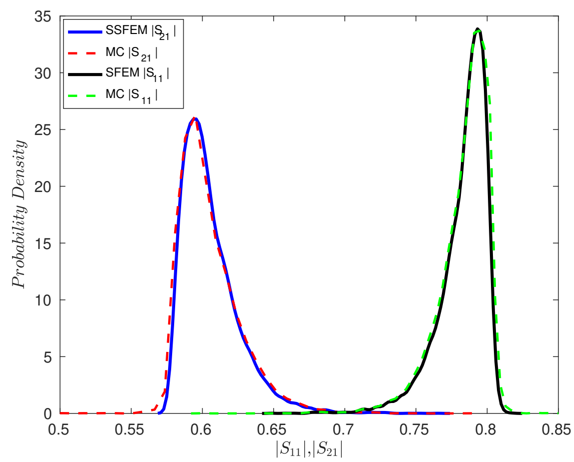

Figure 6. Probability distribution estimation of $|S_{11}|$ and $|S_{21}|$ using lossless dielectrics in the multi layered waveguide model. SSFEM and Monte Carlo with 10000 samples

Monte Carlo method can be used for the stochastic estimation of the probability distributions of transmission and reflection coefficients. Monte Carlo method is impractical even for small number of samples since 3D EM problems contain large number of mesh elements. SSFEM can be used to efficiently predict the probability distribution. In SSFEM, Hermite polynomials are used for numerical implementation of polynomial chaos expansion of the variation in dielectric constant. The SSFEM use KLE to build the matrix system in eq. (11) and solves it only once in contrast to Monte Carlo where the whole process is repeated a large number of times. The Fig. (4) and (5) show a comparison between the probability distribution of the magnitude of transmission coefficient $|S_{21}|$ and reflection coefficient $|S_{11}|$ estimated using SSFEM and Monte Carlo method with different sample sizes. Fig (6) shows the distribution of $|S_{11}|$ and $|S_{21}|$ for the same system without dielectric losses.

The SSFEM results match closely with those predicted using Monte Carlo method with 10000 samples. This is achieved with tremendous saving in the running time. In order to solve the SFEM problem for fixed value of dielectric constant is 18.71 seconds using MATLAB on a 3.6 GHz processor - 32 GB RAM system. A drastic improvement in the execution time has been observed when using SSFEM, over Monte Carlo method for same accuracy. Average execution time for the above problem using SSFEM is 33.24 seconds which is less than twice the runtime required for deterministic FEM, while an an intrusive Monte Carlo method took 189.62 seconds for 100 samples and 13178.79 seconds for 10000 samples.

4 Conclusions

The spectral stochastic finite element formulation for edge element based electromagnetic problems to estimate the stochastic variation of S-parameters with respect to the permittivity variations arising due to the fabrication and environmental tolerances, over multiple subdomains, is presented. Application of SSFEM on a full 3D waveguide problem with material variation is demonstrated by the numerical analysis of a waveguide filled with multiple layers of dielectrics with stochastically varying complex permittivity. The results are validated by comparing the distribution obtained with SSFEM with the output of Monte Carlo analysis. The SSFEM computes the probability distribution with variations of material properties over discrete subdomains highly efficiently, within twice the runtime for a single run of the deterministic problem, while the Monte Carlo simulation took several times the runtime of a sigle run, for the same level of accuracy. SSFEM can therefore be applied to microwave circuits involving various uncertainties over multiple subdomains.

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