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Scattering of Quasi-Electrostatic Waves by a Conducting Cylinder in Hyperbolic Media

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Abstract

In this paper, scattering of quasi-electrostatic waves, propagating in the resonance cone direction, by an infinitely long conducting right circular cylinder in a hyperbolic medium is considered. This problem is analyzed within the quasi-electrostatic approximation, i.e., both incident and scattered waves are represented with the electrostatic potential. The expressions for the electric field components of the scattered field, the surface charge density, the radar cross section per unit length, and the directivity pattern are obtained and analyzed. The results may be important for electrodynamics of magnetoplasmas and metamaterials.

1 Introduction

A classical problem of scattering of electromagnetic waves by a cylinder in isotropic media was solved long ago [1–3]. For about 60 years, this problem has attracted attention of researchers in case of dielectrically anisotropic media due to studies of waves in magnetoplasmas [4] and, more recently, in metamaterials [5]. A very good explanation of the theory of radiation and scattering in anisotropic media is found in [6], and many papers deal with some problems concerning this issue [7–12].

In this paper, we analyze scattering in a homogeneous hyperbolic medium with the dielectric tensor

$$\hat{\varepsilon} = \varepsilon_0 \begin{pmatrix} \varepsilon_\perp & i\varepsilon_\times & 0 \\ -i\varepsilon_\times & \varepsilon_\perp & 0 \\ 0 & 0 & \varepsilon_\parallel \end{pmatrix} \tag{1}$$

where the third coordinate corresponds to the anisotropy axis, ε_0 is the vacuum permittivity, values $\varepsilon_\perp$, $\varepsilon_\parallel$, and $\varepsilon_\times$ are real, and $\varepsilon_\perp \varepsilon_\parallel < 0$. Permeability is supposed to be isotropic and equal to the vacuum permeability μ_0. The iso-frequency surfaces $\omega(\vec{k}) = \text{const}$ corresponding to (1) are shown in Fig. 1 where $k_\perp$ and $k_\parallel$ are the transverse and longitudinal (with respect to the anisotropy axis) components of the wave vector $\vec{k}$. These surfaces are determined by the dispersion relation

$$\left(\frac{\omega}{c}\right)^2 = \frac{k_\perp^2}{\varepsilon_\parallel} + \frac{k_\parallel^2}{\varepsilon_\perp} \tag{2}$$

where c is the speed of light in a vacuum and $\omega/(2\pi)$ is the radiation frequency. The so-called resonance cone direction exists here at angle $\theta_{\text{res}} = \arctan\left(|\varepsilon_\parallel/\varepsilon_\perp|^{1/2}\right)$. Near this direction, the wave numbers k can be arbitrarily large, i.e., $k \gg k_{\text{em}}$ (see Fig. 1), and the waves propagating close to the resonance cone are quasi-electrostatic. Their electric field $\Re[\vec{E}(\vec{r})\exp(-i\omega t)]$ can be described with an electrostatic potential at each moment of time, i.e., $\vec{E}(\vec{r}) = -\nabla\Phi(\vec{r})$, and, in the absence of sources,

$$\text{div}(\hat{\varepsilon}\nabla\Phi) = 0. \tag{3}$$

The magnetic field $\vec{H} = \text{curl}\vec{A}$ of quasi-electrostatic waves can be described using the Coulomb gauge ($\text{div}\vec{A} = 0$). Thus $\nabla^2\vec{A} = i\omega\hat{\varepsilon}\vec{E}$, and this equation can be solved using the theory of Newtonian potential if solution of (3) is known. However, due to the quasi-electrostatic nature of the problem, the magnetic field here is relatively weak ($c\mu_0|\vec{H}| \ll |\vec{E}|$), and thus we only analyze the electric field in this paper.

The hyperbolic metamaterials attract much attention due to their unique optical properties allowing negative refraction, hyperlensing, and cloaking phenomena (see [13–16] and the references in [17]). As to the near-Earth plasma, recent satellite data show [18] that chorus emissions in the radiation belts can propagate in a quasi-electrostatic mode, i.e., close to the resonance cone. Thus study of the electromagnetic properties of hyperbolic media is interdisciplinary. In the present paper, we analyze scattering of quasi-electrostatic waves by an infinitely long conducting right circular cylinder in a homogeneous hyperbolic medium.

2 Equation for Potential: the General Solution

Let us introduce the cylindrical coordinates (ρ, φ, z) where z-axis is directed along the anisotropy axis. Equation (3), written down for potential Φ^{s} of the scattered field $\vec{E}^{\text{s}} = -\nabla\Phi^{\text{s}}$ in a homogeneous medium, then becomes

$$\frac{\partial^2\Phi^{\text{s}}}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\Phi^{\text{s}}}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2\Phi^{\text{s}}}{\partial\varphi^2} - \gamma^2\frac{\partial^2\Phi^{\text{s}}}{\partial z^2} = 0 \tag{4}$$

where $\gamma = |\varepsilon_\parallel/\varepsilon_\perp|^{1/2} = \tan\theta_{\text{res}} > 0$. This equation is hyperbolic, and the off-diagonal components $\pm i\varepsilon_\times$ of tensor (1) do not enter equation (4) because a medium is homogeneous and $\hat{\varepsilon}$ is a Hermitian tensor.

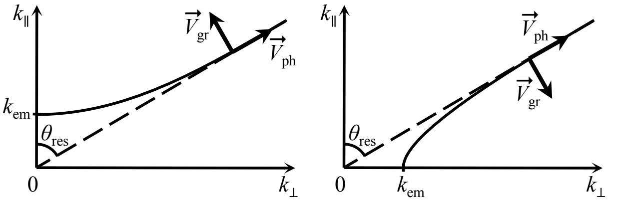

Figure 1. Iso-frequency surfaces $\omega(\vec{k}) = \text{const}$ for $\varepsilon_\perp > 0$, $\varepsilon_\parallel < 0$ (left) and $\varepsilon_\perp < 0$, $\varepsilon_\parallel > 0$ (right). The dashed lines denote the resonance cone.

Firstly, we apply the 1D Fourier transform to the unknown function $\Phi^{\rm s}$:

$$\Phi^{\rm s}(\rho,\varphi,z) = \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty} \bar{\Phi}^{\rm s}(\rho,\varphi,k_z)\exp(ik_z z)\mathrm{d}k_z. \qquad (5)$$

Thus equation (4) becomes

$$\frac{\partial^2\bar{\Phi}^{\rm s}}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\bar{\Phi}^{\rm s}}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2\bar{\Phi}^{\rm s}}{\partial\varphi^2} + \gamma^2 k_z^2\bar{\Phi}^{\rm s} = 0. \qquad (6)$$

Separation of variables in this equation gives

$$\bar{\Phi}^{\rm s}(\rho,\varphi,k_z) = \sum_{m=-\infty}^{+\infty} h_m(k_z)H_m\big(\gamma|k_z|\rho\big)\exp(im\varphi) \qquad (7)$$

where $h_m(k_z)$ is an arbitrary coefficient ($m = 0, \pm1, \pm2, \ldots$),

$$H_m(u) = \begin{cases} H_m^{(1)}(u), & \text{if } \varepsilon_\perp < 0 \text{ and } \varepsilon_\parallel > 0; \\ H_m^{(2)}(u), & \text{if } \varepsilon_\perp > 0 \text{ and } \varepsilon_\parallel < 0, \end{cases}$$

and $H_m^{(1)}(u)$ and $H_m^{(2)}(u)$ are the Hankel functions of the first and second kind of order m, respectively.

The reason why different Hankel functions (depending on $\operatorname{sgn}\varepsilon_\perp$ and $\operatorname{sgn}\varepsilon_\parallel$) enter (7) is that this solution should correspond to the outgoing (from $\rho = 0$ to $\rho \to +\infty$) waves of energy. Felsen [6, 19] and others pointed out that directions of phase and energy propagation in anisotropic media are different. Indeed, the phase propagates in the direction of the wave vector $\vec{k}$ and phase velocity $\vec{V}_{\rm ph} = \omega\vec{k}/|\vec{k}|^2$ whereas the energy propagates in the direction of the Poynting vector and the group velocity $\vec{V}_{\rm gr} = \partial\omega/\partial\vec{k}$. This fact was used, for example, in [9, 10] for the wave field calculations in anisotropic plasmas. As it is widely known (see, e.g., [20]) and follows from Fig. 1, the energy and phase of quasi-electrostatic waves propagate perpendicularly to each other. Limiting ourselves within the transverse (with respect to the anisotropy axis) direction, we conclude from Fig. 1 that energy and phase propagate (a) in opposite directions if $\varepsilon_\perp > 0$ and $\varepsilon_\parallel < 0$ and (b) in one direction if $\varepsilon_\perp < 0$ and $\varepsilon_\parallel > 0$. Since

$$H_m^{(1)}(u) \approx \sqrt{\frac{2}{\pi u}}\exp\Big(iu - i\frac{\pi m}{2} - i\frac{\pi}{4}\Big), \qquad (8)$$

$$H_m^{(2)}(u) \approx \sqrt{\frac{2}{\pi u}}\exp\Big(-iu + i\frac{\pi m}{2} + i\frac{\pi}{4}\Big) \qquad (9)$$

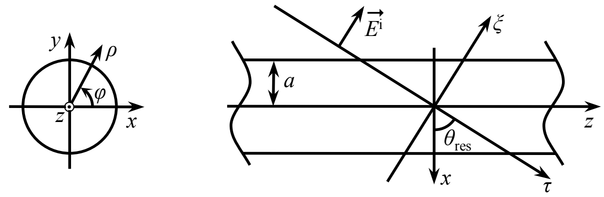

Figure 2. Geometry of the problem.

for $u \gg 1$ [21] and the time convention is selected as $\exp(-i\omega t)$, solution (7) indeed corresponds to the outgoing waves of energy.

The resulting expression (5) with (7) being substituted into the integral contains factor $\exp(\pm i\gamma|k_z|\rho + ik_z z)$ if $\rho \gg 1/(\gamma|k_z|)$ [see the asymptotic expansions (8) and (9)]. Thus the constant phase lines for the scattered field are $z \pm \rho\tan\theta_{\rm res} = \text{const}$ (the sign in this relation depends on the particular problem), and this is in agreement with Fig. 1. The wave energy propagates along these lines, and the phase propagates in a perpendicular direction.

3 The Case of a Plane Quasi-Electrostatic Wave

In this section, we find a partial solution of (4) for the case of scattering of a plane quasi-electrostatic wave by an infinitely long perfectly conducting right circular cylinder of radius a. We assume that the cylinder axis coincides with z-axis so $\rho = a$ is the boundary between the cylinder and background medium (see Fig. 2). To specify the electric field of the incident wave, we introduce the new Cartesian coordinates (τ, y, ξ):

$$\tau = x\cos\theta_{\rm res} + z\sin\theta_{\rm res} = \rho\cos\varphi\cos\theta_{\rm res} + z\sin\theta_{\rm res},$$
$$\xi = -x\sin\theta_{\rm res} + z\cos\theta_{\rm res} = -\rho\cos\varphi\sin\theta_{\rm res} + z\cos\theta_{\rm res}.$$

The new τ- and ξ-axes are obtained using a rotation of x- and z-axes to the resonance angle $\theta_{\rm res}$; y-axis remains unchanged (see Fig. 2).

Taking into account that quasi-electrostatic waves are quasi-longitudinal ($\vec{k} \parallel \vec{E}$), we specify the electric field of the plane incident wave as $\vec{E}^{\rm i}(\vec{r}) = E_0\vec{\xi}_0\exp(ik_0\xi)$ where E_0 is the amplitude, $\vec{\xi}_0$ is a unit vector in the direction of ξ-axis, and $k_0 \gg k_{\rm em}$ is the wave number. The corresponding electrostatic potential then equals $\Phi^{\rm i}(\vec{r}) = (i/k_0)E_0\exp(ik_0\xi)$, or

$$\Phi^{\rm i}(\vec{r}) = \frac{i}{k_0}E_0 e^{ik_0 z\cos\theta_{\rm res}} e^{-ik_0\rho\sin\theta_{\rm res}\cos\varphi}. \qquad (10)$$

In this case, the unknown coefficients h_m can be easily found from the PEC boundary condition, and we come to the following expressions for potential $\Phi^{\rm s}$ and the electric field components $E^{\rm s}_\rho$, $E^{\rm s}_\varphi$, $E^{\rm s}_z$:

$$\begin{pmatrix}\Phi^{\rm s}\\ E^{\rm s}_\rho\\ E^{\rm s}_\varphi\\ E^{\rm s}_z\end{pmatrix} = e^{ik_0 z\cos\theta_{\rm res}} \sum_{m=-\infty}^{+\infty} \beta_m \begin{pmatrix}F^{(\Phi)}_m(\rho)\\ F^{(\rho)}_m(\rho)\\ F^{(\varphi)}_m(\rho)\\ F^{(z)}_m(\rho)\end{pmatrix} e^{im\varphi} \quad (11)$$

where

$$\beta_m = (-i)^{m+1}\frac{J_m(k_0 a\sin\theta_{\rm res})}{H_m(k_0 a\sin\theta_{\rm res})}E_0, \quad (12)$$

$$F^{(\Phi)}_m(\rho) = \frac{1}{k_0}H_m(k_0\rho\sin\theta_{\rm res}), \quad (13)$$

$$F^{(\rho)}_m(\rho) = \frac{m}{k_0\rho}H_m(k_0\rho\sin\theta_{\rm res})- -\sin\theta_{\rm res}H_{m-1}(k_0\rho\sin\theta_{\rm res}), \quad (14)$$

$$F^{(\varphi)}_m(\rho) = -\frac{im}{k_0\rho}H_m(k_0\rho\sin\theta_{\rm res}), \quad (15)$$

$$F^{(z)}_m(\rho) = -i\cos\theta_{\rm res}H_m(k_0\rho\sin\theta_{\rm res}), \quad (16)$$

and $J_m(u)$ is the Bessel function of the first kind of order m. The calculated distribution of potential $\Phi^{\rm s}(\rho,\varphi=0,z)$ of the scattered wave is shown in Fig. 3. The parameters used for calculations are $\varepsilon_\perp = 1$, $\varepsilon_\parallel = -3$ (so $\theta_{\rm res} = 60°$), $a =$ 1 mm, and $k_0 a\sin\theta_{\rm res} = 1$. As one can see from Fig. 3, the phase propagates perpendicularly to lines $z-\rho\tan\theta_{\rm res} =$ const, and this is in agreement with Fig. 1.

The boundary condition for the normal component of the total electric displacement field allows us to find the surface charge density:

$$\sigma(\varphi,z) = \varepsilon_0\left[\sigma^{\rm s}(\varphi)+\sigma^{\rm i}(\varphi)\right]\exp(ik_0 z\cos\theta_{\rm res}) \quad (17)$$

where

$$\sigma^{\rm s}(\varphi) = \sum_{m=-\infty}^{+\infty}\beta_m F^{(\sigma)}_m(\varphi)\exp(im\varphi), \quad (18)$$

$$F^{(\sigma)}_m(\varphi) = \frac{m}{k_0 a}(\varepsilon_\perp+\varepsilon_\times)H_m(k_0 a\sin\theta_{\rm res})- -\varepsilon_\perp\sin\theta_{\rm res}H_{m-1}(k_0 a\sin\theta_{\rm res}), \quad (19)$$

$$\sigma^{\rm i}(\varphi) = (i\varepsilon_\times\sin\varphi-\varepsilon_\perp\cos\varphi)E_0\times \times\sin\theta_{\rm res}\exp(-ik_0 a\sin\theta_{\rm res}\cos\varphi). \quad (20)$$

When $k_0 a\sin\theta_{\rm res} \ll 1$ (i.e., the cylinder is thin), only terms with $m = 0$ survive in expansions (11) and (18).

We note that forms of expressions for the electric field components $E^{\rm s}_\rho$, $E^{\rm s}_\varphi$, $E^{\rm s}_z$ are the same as in the corresponding scattering problem in isotropic media [3]. The effects specific for a hyperbolic medium become apparent in the radial function $H_m(k_0\rho\sin\theta_{\rm res})$, i.e., the Hankel function of either first or second kind depending on $\operatorname{sgn}\varepsilon_\perp$ and $\operatorname{sgn}\varepsilon_\parallel$. In addition to this, the off-diagonal dielectric tensor components appear in the expression for the surface charge density. We also note that, in this problem, it is easier to define the directivity pattern D and the radar cross section per unit length $\mathrm{RCS_{2D}}$ using the electrostatic potential in the far zone ($k_0\rho \to +\infty$) because it allows to obtain the expressions very similar to the isotropic case [3] though the common definition of D and $\mathrm{RCS_{2D}}$ is done with the Poynting vector, electric field, or magnetic field [3]. Consequently, we have

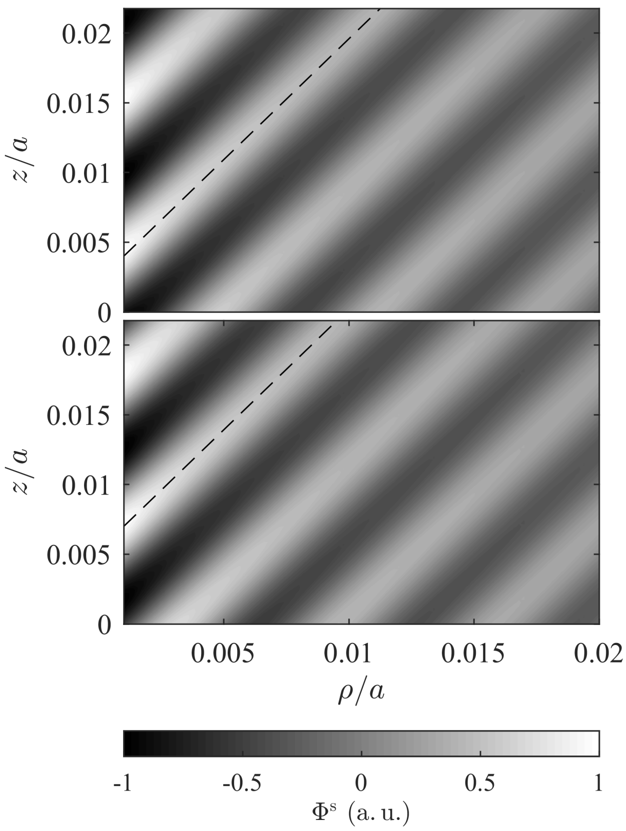

Figure 3. Distributions of real (top) and imaginary (bottom) parts of potential $\Phi^{\rm s}(\rho,\varphi=0,z)$ of the scattered wave. The dashed lines correspond to relation $z-\rho\tan\theta_{\rm res} =$ const.

$$D(\varphi) = \frac{d(\varphi)}{\max[d(\varphi)]}, \quad (21)$$

$$k_0\,\mathrm{RCS_{2D}}(\varphi) = \frac{4}{\sin\theta_{\rm res}}\left|\sum_{m=-\infty}^{+\infty}b_m\exp(im\varphi)\right|^2 \quad (22)$$

where

$$d(\varphi) = \left|\sum_{m=-\infty}^{+\infty}b_m\exp(im\varphi)\right|, \quad (23)$$

$$b_m = (-i)^{m+1}\frac{J_m(k_0 a\sin\theta_{\rm res})}{H_m(k_0 a\sin\theta_{\rm res})}\exp\left(i\frac{\pi m}{2}\operatorname{sgn}\varepsilon_\perp\right). \quad (24)$$

4 The Case of a Quasi-Electrostatic Wave Packet

Since the harmonics of quasi-electrostatic waves (with different k) propagate in one direction (along the resonance cone), it is a natural property of these waves that they have a continuous and relatively wide spectrum of the wave numbers at the single frequency ω [20]. Assuming that each harmonic of this packet has a form of (10), we have

$$h_m(k_z) = -\frac{1}{2\pi H_m(\gamma|k_z|a)} \times \int_{-\infty}^{+\infty}\int_{0}^{2\pi} \Phi^{\rm i}(\rho = a, \varphi', z') \exp(-ik_z z' - im\varphi') {\rm d}z' {\rm d}\varphi', \quad (25)$$

and the resulting expression for potential $\Phi^{\rm s}(\rho,\varphi,z)$ of the scattered wave is determined by relations (5) and (7).

As one can see from (5), (7), and (25), potential $\Phi^{\rm s}(\rho,\varphi,z)$ is generally difficult to calculate. However, if $\max(k_{\max},\Delta) \ll 1/(a\sin\theta_{\rm res})$, then the thin-cylinder approximation may be used. (Here, $k_{\max}$ corresponds to the maximum of the incident wave spectrum and Δ is the characteristic scale, or width, of this spectrum.) In this case, only terms with $m = 0$ survive in the series, and, in certain cases, integration over k_z can be performed analytically using asymptotic expansions.

5 Acknowledgements

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