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Electromagnetic Waves Guided by an Array of Parallel Gyrotropic Cylinders

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Abstract

The propagation of electromagnetic waves guided by an array of parallel identical cylinders that are filled with a gyrotropic medium and placed in free space is studied. It is assumed that the gyrotropic medium is a cold magnetoplasma and the cylinders are aligned with an external static magnetic field. It is shown that a guiding structure consisting of several identical gyrotropic cylinders supports the propagation of a greater number of modes than a single-cylinder waveguide. Conditions are found under which the mutual influence of the cylinders can be neglected when calculating the properties of the guided modes.

1 Introduction

The recent decades have shown an enhanced interest in the excitation and propagation of nonsymmetric electromagnetic waves supported by arrays of open waveguides filled with nonreciprocal media. Such guiding structures can play an important role in many applications ranging from the laboratory radio-frequency plasma sources [1] to optical transmission systems [2]. The features of propagation of eigenmodes of the corresponding waveguides are of primary importance for developing and advancing such applications. Although the propagation features of eigenmodes of individual open cylindrical gyrotropic waveguides are well studied [3, 4], the properties of modes of a complex waveguide consisting of a few such guiding structures have poorly been presented in the literature.

This work is devoted to a study of the dispersion properties of modes of a system that consists of identical gyrotropic cylinders located in free space.

2 Formulation of the Problem and Basic Equations

Consider an array of identical, infinitely long circular cylinders of radius a, which are located in free space and filled with a gyrotropic medium, namely, a cold magnetoplasma (see Fig. 1). The cylinders are parallel to the z axis of a Cartesian coordinate system (x,y,z) and are aligned with an external static magnetic field $\mathbf{B}_0$. In this case, the plasma is described by the permittivity tensor

$$\hat{\varepsilon} = \begin{pmatrix} \varepsilon & -ig & 0 \\ ig & \varepsilon & 0 \\ 0 & 0 & \eta \end{pmatrix}. \quad (1)$$

Here,

$$\varepsilon = 1 - \frac{\omega_{\rm p}^2(\omega - i\nu_{\rm e})}{[(\omega - i\nu_{\rm e})^2 - \omega_{\rm H}^2]\omega}, \quad \eta = 1 - \frac{\omega_{\rm p}^2}{\omega(\omega - i\nu_{\rm e})},$$
$$g = \frac{\omega_{\rm p}^2\omega_{\rm H}}{[(\omega - i\nu_{\rm e})^2 - \omega_{\rm H}^2]\omega}, \quad (2)$$

where $\omega_{\rm H}$ is the electron gyrofrequency, $\omega_{\rm p}$ is the electron plasma frequency, and $\nu_{\rm e} = \nu_{\rm ei} + \nu_{\rm en}$ is the sum of the electron–ion and electron–neutral collision frequencies in the plasma. In the tensor elements, we neglected the contribution of ions. This is possible under the condition $|\omega - i\nu_{\rm e}| \gg \omega_{\rm LH}$ [4], which is assumed throughout this work (here, $\omega_{\rm LH}$ is the lower hybrid resonance frequency). The Gaussian system of units is used in this paper.

The field in the coordinate system of the jth cylinder, with the $\exp(i\omega t)$ time dependence dropped, can be represented

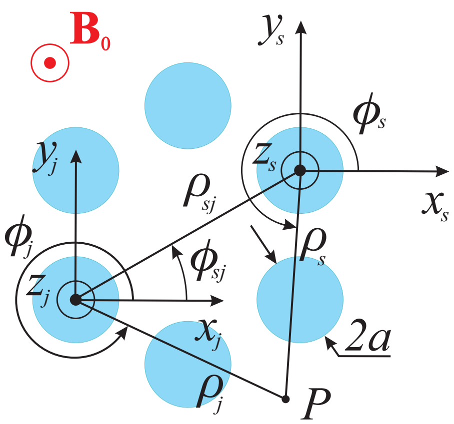

Figure 1. Geometry of the problem. Observation point P with its radial and azimuthal coordinates, $(\rho_j,\ \phi_j)$ and $(\rho_s,\ \phi_s)$, in the coordinate systems related to the jth and the sth cylinders, respectively.

in terms of azimuthal harmonics as

$$\mathbf{E}_j = \sum_{m=-\infty}^{\infty} \mathbf{E}_{j,m} \exp[-i(m\phi_j + k_0 pz)],$$
$$\mathbf{H}_j = \sum_{m=-\infty}^{\infty} \mathbf{H}_{j,m} \exp[-i(m\phi_j + k_0 pz)], \qquad (3)$$

where m is the azimuthal index ($m = 0, \pm 1, \pm 2, \ldots$), k_0 is the wave number in free space, and p is the normalized (to k_0) longitudinal propagation constant. In turn, the vector quantities $\mathbf{E}_{j,m}$ and $\mathbf{H}_{j,m}$ can be expressed via their longitudinal components $E_{z;j,m}$ and $H_{z;j,m}$ that satisfy the following equations in the plasma medium [4]:

$$\hat{L}_m E_{z;j,m} - k_0^2 \frac{\eta}{\varepsilon}(p^2 - \varepsilon) E_{z;j,m} = -ik_0^2 \frac{g}{\varepsilon} p H_{z;j,m},$$
$$\hat{L}_m H_{z;j,m} - k_0^2 \left(p^2 + \frac{g^2}{\varepsilon} - \varepsilon\right) H_{z;j,m} = ik_0^2 \frac{g}{\varepsilon} \eta p E_{z;j,m}, \quad (4)$$

where $\hat{L}_m = \dfrac{d^2}{d\rho_j^2} + \dfrac{1}{\rho_j}\dfrac{d}{d\rho_j} - \dfrac{m^2}{\rho_j^2}$. The transverse field components $E_{\rho;j,m}$, $E_{\phi;j,m}$, $H_{\rho;j,m}$, and $H_{\phi;j,m}$ can be expressed via $E_{z;j,m}$ and $H_{z;j,m}$ (see [4]).

The azimuthal harmonics of the longitudinal field components inside the jth cylinder are represented in the form [4]

$$E^{(c)}_{z;j,m} = \frac{i}{\eta} \sum_{k=1}^{2} B^{(k)}_{j,m} n_k q_k J_m(k_0 q_k \rho_j),$$
$$H^{(c)}_{z;j,m} = -\sum_{k=1}^{2} B^{(k)}_{j,m} q_k J_m(k_0 q_k \rho_j). \qquad (5)$$

Here, the superscript (c) denotes the field inside the cylinder, J_m is a Bessel function of the first kind of order m, $B^{(1)}_{j,m}$ and $B^{(2)}_{j,m}$ are the amplitude coefficients corresponding to the azimuthal index m, and

$$q_k^2(p) = \Big\{ \varepsilon^2 - g^2 + \varepsilon\eta - (\varepsilon+\eta)p^2 + (-1)^k \Big[(\varepsilon-\eta)^2 p^4$$
$$+ 2\big(g^2(\varepsilon+\eta) - \varepsilon(\varepsilon-\eta)^2\big)p^2 + (\varepsilon^2 - g^2 - \varepsilon\eta)^2 \Big]^{1/2} \Big\} / (2\varepsilon),$$
$$n_k = -\frac{\varepsilon}{pg}\left(p^2 + q_k^2 + \frac{g^2}{\varepsilon} - \varepsilon\right), \quad k = 1, 2. \qquad (6)$$

The presence of two transverse wave numbers q_1 and q_2 in the magnetized plasma medium, which correspond to the same longitudinal wave number p, is related to the anisotropic properties of a magnetoplasma.

The field outside the jth cylinder is represented as a superposition of the field scattered by the jth cylinder and the field incident on this cylinder from other cylinders. The scattered field, which is denoted by the superscript (s), is also written in terms of cylindrical functions and has the following longitudinal components:

$$E^{(s)}_{z;j,m} = C^{(1)}_{j,m} q H^{(2)}_m(k_0 q \rho_j),$$
$$H^{(s)}_{z;j,m} = C^{(2)}_{j,m} q H^{(2)}_m(k_0 q \rho_j), \qquad (7)$$

where $H^{(2)}_m$ is a Hankel function of the second kind of order m, $C^{(1)}_m$ and $C^{(2)}_m$ are the scattering coefficients corresponding to the azimuthal index m, and q is the normalized (to k_0) transverse wave number in the outer region, which is written as $q = (1 - p^2)^{1/2}$.

The azimuthal harmonics of the field incident on the jth cylinder from, e.g., the sth cylinder can be rewritten in the coordinate system related to the jth cylinder with the help of Graf's addition theorem for cylindrical functions [5]:

$$H^{(2)}_m(k_0 q \rho_s)\, e^{-im\phi_s}$$
$$= \sum_{n=-\infty}^{\infty} J_n(k_0 q \rho_j)\, H^{(2)}_{n-m}(k_0 q \rho_{sj})\, e^{i(n-m)\phi_{sj} - in\phi_j}, \quad (8)$$

where the condition $\rho_j < \rho_{sj}$ should be satisfied. In what follows, this condition is ensured due to the fact that Eq. (8) is used only for representing the incident field on the surface of each cylinder, i.e., at $\rho_j = a < \rho_{sj}$. Other notations in Eq. (8) are clarified in Fig. 1. Hence, the azimuthal harmonics of the longitudinal field components in the outer region in the coordinate system of the jth cylinder are represented in the form

$$E_{z;j,m} = E^{(s)}_{z;j,m} + E^{(i)}_{z;j,m}, \quad H_{z;j,m} = H^{(s)}_{z;j,m} + H^{(i)}_{z;j,m}, \qquad (9)$$

where

$$E^{(i)}_{z;j,m} = \mathscr{C}^{(1)}_{j,m} q J_m(k_0 q \rho_j), \qquad (10)$$
$$H^{(i)}_{z;j,m} = \mathscr{C}^{(2)}_{j,m} q J_m(k_0 q \rho_j), \qquad (11)$$
$$\mathscr{C}^{(k)}_{j,m} = \sum_{s \neq j}^{N_c} \sum_{n=-\infty}^{\infty} C^{(k)}_{s,n} H^{(2)}_{m-n}(k_0 q \rho_{sj})\, e^{i(m-n)\phi_{sj}}. \qquad (12)$$

Here, N_c is the total number of the cylinders, and the summation with respect to s is performed over all the cylinders, excepting the jth one. The terms (10) and (11) can be interpreted as the longitudinal components of the field of a cylindrical wave that is incident on the jth cylinder. The continuity conditions of the tangential field components at $\rho_j = a$ allows one to obtain the dispersion relation determining the longitudinal wave numbers p of modes which are guided by the array of cylinders. For the jth cylinder, these boundary conditions at $\rho_j = a$ are written as

$$E^{(c)}_{z;j,m} = E^{(s)}_{z;j,m} + E^{(i)}_{z;j,m}, \quad H^{(c)}_{z;j,m} = H^{(s)}_{z;j,m} + H^{(i)}_{z;j,m},$$
$$E^{(c)}_{\phi;j,m} = E^{(s)}_{\phi;j,m} + E^{(i)}_{\phi;j,m}, \quad H^{(c)}_{\phi;j,m} = H^{(s)}_{\phi;j,m} + H^{(i)}_{\phi;j,m}. \quad (13)$$

Application of conditions (13) to each cylinder successively for $m = 0, \pm 1, \pm 2, \ldots, \pm M$ yields a system of $4N_c(2M + 1)$ homogeneous equations for the coefficients $B^{(1)}_{j,m}$, $B^{(2)}_{j,m}$, $C^{(1)}_{j,m}$, and $C^{(2)}_{j,m}$, where M is the absolute value of the number of the highest-order azimuthal harmonic still taken into account. The choice of a finite total number of the azimuthal harmonics is determined by the required accuracy of numerical calculations. The obtained system of $4N_c(2M+1)$ homogeneous equations yields a dispersion relation which allows us to find the longitudinal wave numbers p of modes

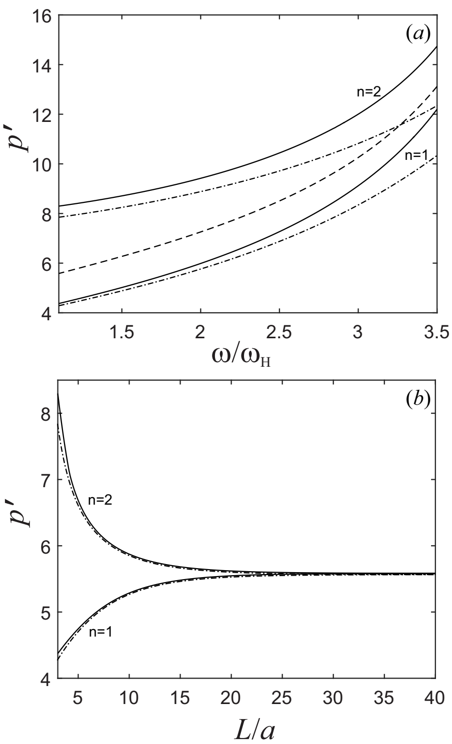

Figure 2. (a) Dispersion curves of modes of a two-cylinder waveguide (solid lines) and a single-cylinder waveguide (dashed line) for $L = 3a$, $\omega_p a/c = 0.19$, and $\omega_p/\omega_H = 6.42$ in the absence of collisional loss (ν_e = 0). (b) Propagation constants of modes of the two-cylinder waveguide as functions of the distance L for $\omega/\omega_H = 1.1$.

of the complex waveguide considered. The dispersion relation is then reduced to the equality of the determinant of such a system to zero. The longitudinal wave numbers are complex ($p = p' - ip''$) due to the presence of collisional loss. Here, p' and p'' are the propagation and damping constants, respectively.

3 Numerical Results

Numerical calculations were performed for the following dimensionless parameters which can be ensured in laboratory experiments in the radio-frequency range [1]: $\omega_p a/c = 0.19$ and $\omega_p/\omega_H = 6.42$. We seek modes with the frequencies lying between $1.1\omega_H$ and $3.5\omega_H$. Note that in this frequency range, a single-cylinder waveguide is electrically thin ($k_0 a \ll 1$) and supports the propagation of one symmetric mode in the considered interval of the propagation constants. Thus, this frequency interval is most suited for a study of the influence of the geometric parameters of the guiding system on the properties of its modes. The maximum absolute value of the azimuthal index was limited to $M = 3$ in our calculations.

3.1 Two-cylinder waveguide

Consider a waveguide consisting of two loss-free plasma cylinders (ν_e = 0), which are located at distance L from each other (in this case, $p = p'$). Figure 2(a) shows the longitudinal wave numbers p as functions of frequency for the modes of such a waveguide (solid lines) at $L = 3a$ and the mode of a single-cylinder waveguide (dashed line). It is seen in this figure that the two-cylinder waveguide supports the propagation of two modes that have similar dependences $p(\omega)$. The behaviour of the dispersion curves of these modes is close to that of the dispersion curve of the single-cylinder waveguide. For a two-cylinder waveguide, the approximate dispersion relation can be obtained in the following form:

$$H_0^{(2)}(k_0 qL) \pm \frac{2H_1^{(2)}(Q) - \eta Q H_0^{(2)}(Q)}{2J_1(Q) - \eta Q J_0(Q)} = 0, \qquad (14)$$

where $Q = k_0 qa$, and the plus and minus signs correspond to the modes with $n = 1$ and $n = 2$, respectively. Hereafter, the number n is the number of the mode in order of increasing propagation constant. The dispersion curves, which are found from approximate relation (14), are shown in Fig. 2 by the dash-dot lines.

An increase in the distance between the cylinders leads to a decrease in the influence of the adjacent cylinder on the fields of the modes. In the limit $L \to \infty$, we arrive at the case of independent single-cylinder waveguides. This fact can be seen in Fig. 2(b) which shows the dependences of the propagation constants p on the distance L, which are found from the rigorous dispersion relation (solid lines) and the approximate dispersion relation (14) (dash-dot lines). In this figure, the propagation constants of the modes tend to the propagation constant of the mode guided by the single cylinder with increasing L. In the case of an electrically small distance L, the approximate dependences of q on L can be obtained from Eq. (14) as follows:

$$L = \mathrm{Re}\left\{\frac{2}{k_0 q} \exp\left[-\gamma - i\frac{\pi}{2}\left(1 \pm \frac{2H_1^{(2)}(Q) - \eta Q H_0^{(2)}(Q)}{2J_1(Q) - \eta Q J_0(Q)}\right)\right]\right\}, \qquad (15)$$

where $\gamma = 0.5772...$ is Euler's constant. Formula (15) gives a good approximation for the behavior of the propagation constant p as a function of L for the relatively small values of L in Fig. 2(b).

3.2 Six-cylinder waveguide

We now consider a waveguide consisting of six magnetized plasma cylinders, which can be realized in experiments [1]. The axes of the cylinders are located on a cylindrical surface of radius L so that the distance between the axes of the nearest cylinders is equal to L (see Fig. 1). Figure 3(a) shows the propagation constants as functions of frequency for the modes of the considered waveguide (solid color lines) at $L = 3a$ and the propagation constant for the eigenmode of the single-cylinder waveguide (dashed line) in the case $\nu_e = 0$. The number of modes which are supported by such a complex waveguide increases compared with the previous case of a two-cylinder waveguide. Figure 3(b) shows

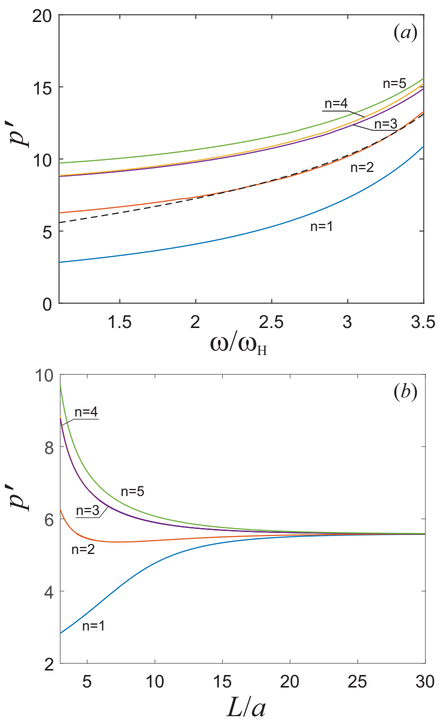

Figure 3. The same as in Fig. 2 but for a six-cylinder waveguide.

the dependences of the quantities p for these modes on L. These dependences are similar to those for modes of the two-cylinder waveguide (Fig. 2(b)). Note that the mode with $n = 2$ has the nonmonotonic dependence of p on L in Fig. 3(b).

The propagation constants (p') and the damping constants (p'') as functions of the electron collision frequency are presented in Fig. 4. These dependences are similar to those of strongly damped modes of the plasma column, which were studied in [3] and whose damping rate can be estimated as $p''/p' \approx \nu_e/\omega$. It is seen in Fig. 4 that allowance for the relatively small collisional loss does not lead to significant changes in the propagation constants.

4 Conclusion

In this work, we have analyzed the properties of modes which are supported by an open waveguide consisting of several identical gyrotropic cylinders filled with a magnetoplasma in the rf range. It is shown that such a complex guiding structure supports the greater number of modes than a single-cylinder waveguide. It is demonstrated that the modes of the complex waveguide transform to the modes of a single-cylinder waveguide with increasing distance between the cylinders in the multi-cylinder structure. It is shown that for the considered parameters, allowance for the relatively small collisional loss in the plasma does not lead to significant changes in the propagation constants of modes of the complex waveguide.

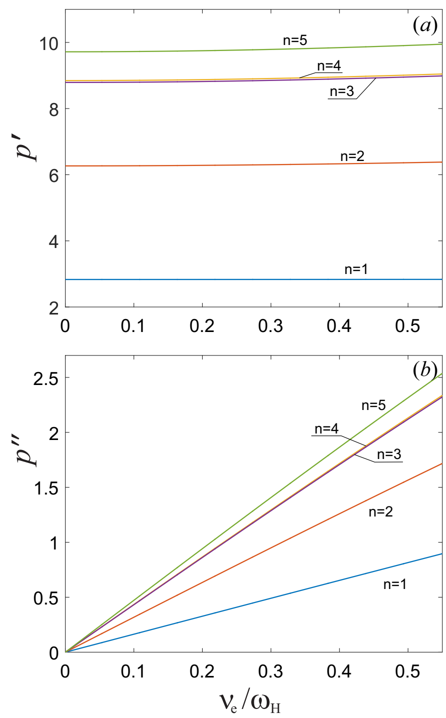

Figure 4. (a) Propagation constants and (b) damping constants of modes of a six-cylinder waveguide as functions of the electron collision frequency ν_e for $\omega_p a/c = 0.19$, $\omega_p/\omega_H = 6.42$, and $\omega/\omega_H = 1.1$.

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