2019 URSI Commission B School for Young Scientists

Field and Potential Based Methods in Anisotropic and Bianisotropic Electromagnetics

Lecture Notes

May 27, 2019

Westin San Diego Hotel

San Diego, CA, USA

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* This School is organized during the "URSI Commission B International Symposium on Electromagnetic Theory" (EMTS 2019), May 27 - 31, 2019, San Diego, CA, USA.

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Preface

The “2019 URSI Commission B School for Young Scientists” is organized by URSI Commission B and is arranged on the occasion of the “URSI Commission B International Symposium on Electromagnetic Theory (EMTS 2019), May 27 - 31, 2019, Westin San Diego Hotel, San Diego, CA, USA. This School is a one-day event held during EMTS 2019, and is sponsored jointly by URSI Commission B and the EMTS 2019 Organizing Committee. The School offers a short, intensive course, where a series of lectures will be delivered by a leading scientist in the Commission B community. Young scientists are encouraged to learn the fundamentals and future directions in the area of electromagnetic theory from these lectures.

Program

1. Course Title

Field and Potential Based Methods in Anisotropic and Bianisotropic Electromagnetics

2. Course Instructor

Prof. Michael J. Havrilla
Department of Electrical and Computer Engineering,
Air Force Institute of Technology (AFIT), Wright-Patterson AFB, OH, USA

3. Course Program

Date: Monday, May 27, 2019
Venue: Westin San Diego Hotel, San Diego, CA, USA (EMTS 2019 venue)
Schedule (Coffee breaks are also included):

0800-0845	Lecture 1	Maxwell's Equations and Constitutive Relations
0900-0945	Lecture 2	Factors that Influence Anisotropy and Bianisotropy
1000-1045	Lecture 3	Field and Potential-Based Methods of Analysis
1100-1145	Lecture 4	Field-Based Examples – Sources Not Present
1200-1300	LUNCH	
1300-1345	Lecture 5	Field-Based Examples – Sources Present
1400-1445	Lecture 6	Potential-Based Examples – Sources Not Present
1500-1545	Lecture 7	Potential-Based Examples – Sources Present
1600-1645	Lecture 8	Conclusion and Future Research

Lecture Abstract

Field and Potential Based Methods in Anisotropic and Bianisotropic Electromagnetics

Prof. Michael J. Havrilla
Department of Electrical and Computer Engineering,
Air Force Institute of Technology (AFIT), Wright-Patterson AFB, OH, USA

Recent advances in rapid prototyping techniques, such as 3D printing, have made the manufacturing of complex media such as anisotropic and bianisotropic media possible. This capability has subsequently placed a greater need to incorporate the teaching of complex media into the advanced undergraduate and graduate educational curricula. The goal of this short course is to develop and demonstrate both field and potential based analytical techniques for the solution of electromagnetic problems involving complex media. First, it will be shown how symmetry has a profound influence on material tensor properties and how this symmetry can be utilized to fabricate anisotropic and bianisotropic media. Next, it will be shown how these material tensor properties influence the method of analysis; either a field-based or potential-based technique. Field-based techniques, which are directly based upon Maxwell's equations, will be discussed first. Examples, including the analysis of plane waves in general bianisotropic media and the analysis of a parallel-plate waveguide filled with a uniaxial medium, will be provided to demonstrate the field-based methodology. It will also be shown why the well-known vector potential method for isotropic media becomes invalid for complex media. This subsequently leads to a scalar potential formulation that is valid for gyrotropic anisotropic or gyrotropic bianisotropic media. Examples of the scalar potential formalism are given, including the analysis of a parallel-plate waveguide filled with a uniaxial medium. A comparison between the field and potential-based formalisms is provided to better understand the advantages and limitations of each method. A conclusion and future recommendations are also provided.

Biographical Sketch of Course Instructor

Michael J. Havrilla received B.S. degrees in Physics and Mathematics in 1987, the M.S.E.E degree in 1989 and the Ph.D. degree in electrical engineering in 2001 from Michigan State University, East Lansing, MI. From 1990-1995, he was with General Electric Aircraft Engines, Evendale, OH and Lockheed Skunk Works, Palmdale, CA, where he worked as an electrical engineer. He is currently a Professor in the Department of Electrical and Computer Engineering at the Air Force Institute of Technology (AFIT), Wright-Patterson AFB, OH. He is a member of URSI Commission B, a senior member of the IEEE, a senior member and current Vice President of the Antenna Measurement Techniques Association (AMTA), and a member of the Eta Kappa Nu and Sigma Xi honor societies. Dr. Havrilla has received various teaching and research awards, including the AFIT Instructor of the Quarter Award and the Air Force John L. McLucas Basic Research Award. His current research interests include electromagnetic and guided-wave theory, electromagnetic propagation and radiation in anisotropic and bianisotropic materials, electromagnetic characterization of complex media, quantum field theory and general relativity.

Field and Potential Based Methods in Anisotropic and Bianisotropic Electromagnetics

May 27, 2019

Prof. Michael J. Havrilla

Department of Electrical and Computer Engineering,

Air Force Institute of Technology (AFIT),

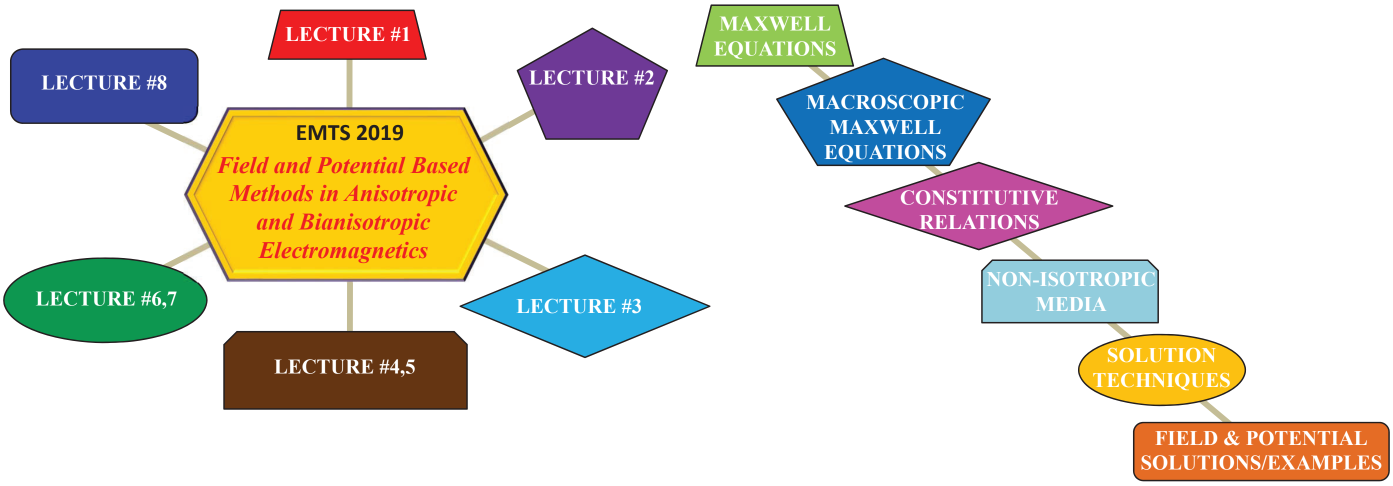
Wright-Patterson AFB, OH, USA

2019 International Symposium on Electromagnetic Theory

Field and Potential Based Methods in Anisotropic and Bianisotropic Electromagnetics

Dr. Michael J. Havrilla
Professor of Electrical Engineering
Air Force Institute of Technology
WPAFB, Ohio 45433

Overview – Lectures/Big Picture



LECTURE #1: Review various forms and regimes of validity of Maxwell's equations and constitutive relations.
LECTURE #2: Discuss factors that influence anisotropy and bianisotropy.
LECTURE #3: Discuss field and potential based methods for solving electromagnetic problems.
LECTURE #4,5: Provide several examples of field based solutions in complex media within and exterior to source regions.
LECTURE #6,7: Provide several examples of potential based solutions in complex media within and exterior to source regions.
LECTURE #8: Summary, conclusions and future research.

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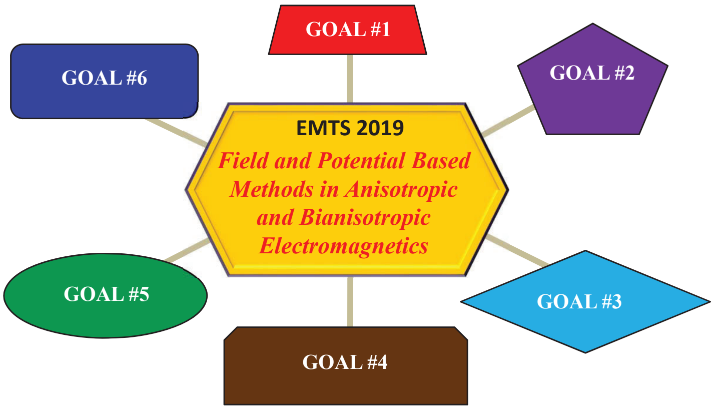
Overview – Primary Goals



GOAL #1: Gain a deeper appreciation of Maxwell's equations and the regimes of validity.

GOAL #2: Develop a better understanding of constitutive relations and recent areas of electromagnetic material research.

GOAL #3: Understand the profound influence that symmetry has on material tensor properties and design.

GOAL #4: Learn how to solve Maxwell's equations involving complex media using field and potential based techniques.

GOAL #5: Obtain deeper physical insight into electromagnetic field behavior in non-isotropic environments.

GOAL #6: Apply knowledge learned in your own personal research.

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2019 International Symposium on Electromagnetic Theory

LECTURE #1
Maxwell Equations and Constitutive Relations

Dr. Michael J. Havrilla
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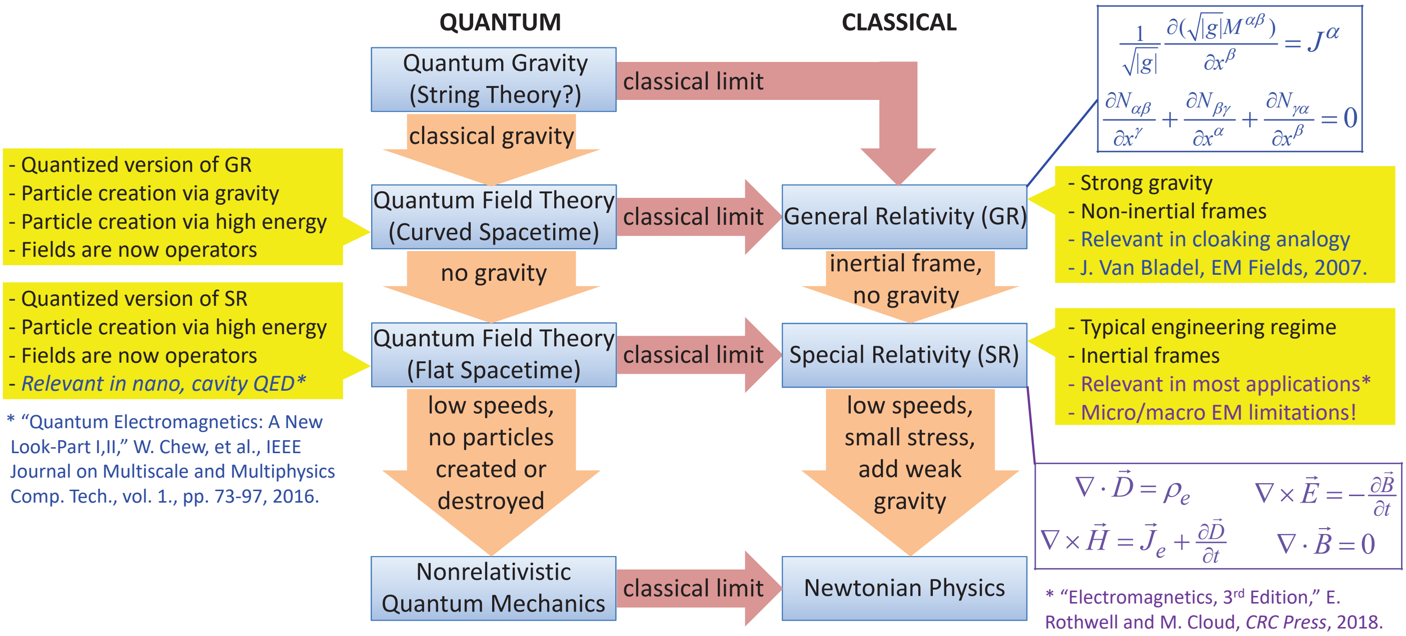
Fundamental Physics and Maxwell Equations (Quantum vs. Classical)*



* "Quantum Electromagnetics: A New Look-Part I,II," W. Chew, et al., IEEE Journal on Multiscale and Multiphysics Comp. Tech., vol. 1., pp. 73-97, 2016.

* "Electromagnetics, 3rd Edition," E. Rothwell and M. Cloud, *CRC Press*, 2018.

* Adopted from "Modern Classical Physics," Kip Thorne and Roger Blandford, Princeton University Press, 2017.

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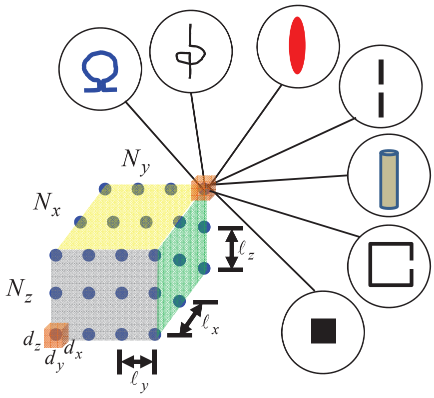
Maxwell Equations (Microscopic vs. Macroscopic Assessment)

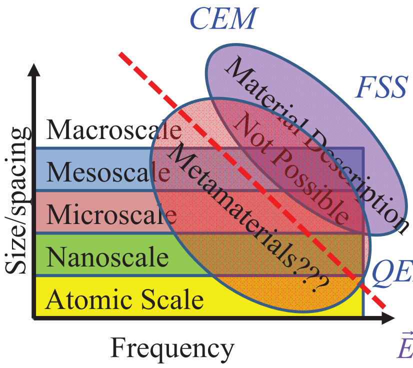
MICROSCOPIC

$$\varepsilon_0 \nabla \cdot \vec{e} = \eta$$
$$\frac{1}{\mu_0}\nabla \times \vec{b} = \vec{i} + \varepsilon_0 \frac{\partial \vec{e}}{\partial t}$$
$$\nabla \times \vec{e} = -\frac{\partial \vec{b}}{\partial t}$$
$$\nabla \cdot \vec{b} = 0$$

$\vec{e}, \vec{b} \ldots$ *microscopic fields*

$\eta, \vec{i} \ldots$ *microscopic charge, current*





MACROSCOPIC

$$\nabla \cdot \vec{D} = \rho_e$$
$$\nabla \times \vec{H} = \vec{J}_e + \frac{\partial \vec{D}}{\partial t}$$
$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$
$$\nabla \cdot \vec{B} = 0$$

$\vec{E}, \vec{B} \ldots$ *macroscopic / average fields*

$\rho_e, \vec{J}_e \ldots$ *macroscopic charge, current*

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P} - \nabla \cdot \bar{\bar{Q}} + \cdots$$
$$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M} + \nabla \cdot \bar{\bar{M}}^Q - \cdots$$

$$\left.\begin{matrix} N_\alpha >> 1 \\ d_\alpha << \lambda \\ \ell_\alpha << \lambda \end{matrix}\right\} \ldots \alpha = x, y, z \ \ (\textit{for valid macroscopic EM model})!!!$$

G. Russakoff, "A Derivation of the Macroscopic Maxwell Equations," American Journal of Physics, Vol. 38, No. 10, pp. 1188 – 1195, 1970.

[1] Adopted from S. Tretyakov, "Contemporary notes on metamaterials," IET Microw. Antennas Propag., 2007, 1, (1), pp. 3–11.
[2] Adopted from A. Sihvola, "Metamaterials: a personal view," Radioengineering, Vol. 18, No. 2, June 2009.
[3] Adopted from A. Sihvola, "Metamaterials in electromagnetics," Metamaterials Vol. 1, No. 1 (2007) 2–11.

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Macroscopic Maxwell Equations (Various Forms - Valid for Inertial Frames)

Vector Form (Engineering / Working Form)

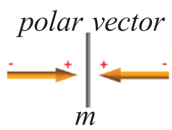
$$\nabla \cdot \vec{D} = \rho_e$$
$$\nabla \times \vec{H} = \vec{J}_e + \frac{\partial \vec{D}}{\partial t}$$
$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$
$$\nabla \cdot \vec{B} = 0$$

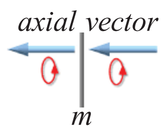
Tensor Form (Manifestly Covariant Form)

$$\frac{\partial F_{\alpha\beta}}{\partial x^\gamma} + \frac{\partial F_{\beta\gamma}}{\partial x^\alpha} + \frac{\partial F_{\gamma\alpha}}{\partial x^\beta} = 0 \ , \ \frac{\partial G^{\alpha\mu}}{\partial x^\alpha} = J^\mu$$

$$F_{\mu\nu} = \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -cB_z & cB_y \\ -E_y & cB_z & 0 & -cB_x \\ -E_z & -cB_y & cB_x & 0 \end{bmatrix} , \ G^{\mu\nu} = \begin{bmatrix} 0 & -cD_x & -cD_y & -cD_z \\ cD_x & 0 & -H_z & H_y \\ cD_y & H_z & 0 & -H_x \\ cD_z & -H_y & H_x & 0 \end{bmatrix}$$

$$x^0 = ct \ , x^1 = x \ , x^2 = y \ , x^3 = z \ ; \ J^0 = c\rho_e \ , J^1 = J_{ex} \ , J^2 = J_{ey} \ , J^3 = J_{ez}$$





Clifford Form (polar / axial vector clarity)

$$\nabla \cdot \vec{D} = \rho_e \ (\textit{scalar})$$
$$\nabla \cdot (\hat{H}) = -\vec{J}_e - \frac{\partial \vec{D}}{\partial t} \ (\textit{vector})$$
$$\nabla \wedge \vec{E} = -\frac{\partial \hat{B}}{\partial t} \ (\textit{bivector / pseudovector})$$
$$\nabla \wedge \hat{B} = 0 \ (\textit{trivector / pseudoscalar})$$

or

$$\bar{\nabla} \wedge \hat{F} = 0 \ , \ \bar{\nabla} \cdot \hat{G} = \bar{J}$$
$$\bar{\nabla} = \nabla - \hat{e}_0 \frac{\partial}{\partial ct} \ , \ \hat{F} = \vec{E}\hat{e}_0 - c\hat{B} \ , \ \hat{G} = c\vec{D}\hat{e}_0 - \hat{H} \ , \ \bar{J} = \hat{e}_0 c\rho_e + \vec{J}_e$$
$$\hat{B} = I\vec{B} \ , \ \hat{H} = I\vec{H} \ , \ I = \hat{e}_1\hat{e}_2\hat{e}_3 = \hat{x}\hat{y}\hat{z}$$

Pertti Lounesto, "Clifford Algebras and Spinors, Second Edition," Cambridge University Press, 2001.

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Macroscopic Maxwell Equations (Including Magnetic Sources - Valid for Inertial Frames)

Vector Form

$$\nabla\cdot\vec{D}=\rho_e$$
$$\nabla\times\vec{H}=\vec{J}_e+\frac{\partial\vec{D}}{\partial t}$$
$$\nabla\times\vec{E}=-\vec{J}_h-\frac{\partial\vec{B}}{\partial t}$$
$$\nabla\cdot\vec{B}=\rho_h$$

Tensor Form

$$\frac{\partial F_{\alpha\beta}}{\partial x^\gamma}+\frac{\partial F_{\beta\gamma}}{\partial x^\alpha}+\frac{\partial F_{\gamma\alpha}}{\partial x^\beta}=-M_{\alpha\beta\gamma}\ ,\ \frac{\partial G^{\alpha\mu}}{\partial x^\alpha}=J^\mu$$

$$F_{\mu\nu}=\begin{bmatrix}0 & E_x & E_y & E_z\\ -E_x & 0 & -cB_z & cB_y\\ -E_y & cB_z & 0 & -cB_x\\ -E_z & -cB_y & cB_x & 0\end{bmatrix},\ G^{\mu\nu}=\begin{bmatrix}0 & -cD_x & -cD_y & -cD_z\\ cD_x & 0 & -H_z & H_y\\ cD_y & H_z & 0 & -H_x\\ cD_z & -H_y & H_x & 0\end{bmatrix},$$

$$M_{\alpha\beta\gamma}=0\ (\alpha=\beta,\beta=\gamma,\alpha=\gamma)$$
$$M_{123,231,312}=-M_{321,132,213}=c\rho_h$$
$$M_{023,230,302}=-M_{320,032,203}=-J_{hx}$$
$$M_{031,310,103}=-M_{130,013,301}=-J_{hy}$$
$$M_{012,120,201}=-M_{210,021,102}=-J_{hz}$$

$$x^0=ct\ ,\ x^1=x\ ,\ x^2=y\ ,\ x^3=z\ \ ;\ \ J^0=c\rho_e\ ,\ J^1=J_{ex}\ ,\ J^2=J_{ey}\ ,\ J^3=J_{ez}$$

Clifford Form

$$\nabla\cdot\vec{D}=\rho_e\ (scalar)$$
$$\nabla\cdot(\hat{H})=-\vec{J}_e-\frac{\partial\vec{D}}{\partial t}\ (vector)$$
$$\nabla\wedge\vec{E}=-\hat{J}_h-\frac{\partial\hat{B}}{\partial t}\ (bivector\,/\,pseudovector)$$
$$\nabla\wedge\hat{B}=\bar{\rho}_h\ (trivector\,/\,pseudoscalar)$$

or

$$\bar{\nabla}\wedge\hat{F}=-\bar{M}\ ,\ \bar{\nabla}\cdot\hat{G}=\bar{J}$$
$$\bar{\nabla}=\nabla-\hat{e}_0\frac{\partial}{\partial ct}\ ,\ \hat{F}=\vec{E}\hat{e}_0-c\hat{B}\ ,\ \hat{G}=c\vec{D}\hat{e}_0-\hat{H}$$
$$\bar{J}=\hat{e}_0 c\rho_e+\vec{J}_e\ ,\ \bar{M}=c\bar{\rho}_h+\hat{J}_h\hat{e}_0$$
$$\hat{B}=I\vec{B}\ ,\ \hat{H}=I\vec{H}\ ,\ \hat{J}_h=I\vec{J}_h\ ,\ \bar{\rho}_h=I\rho_h\ ,\ I=\hat{e}_1\hat{e}_2\hat{e}_3=\hat{x}\hat{y}\hat{z}$$

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Macroscopic Maxwell Equations (Vector Form) – Transform Domain

$$\nabla\times\vec{E}(\vec{r},t)=-\vec{J}_h(\vec{r},t)-\frac{\partial\vec{B}(\vec{r},t)}{\partial t}\ ,\ \nabla\cdot\vec{D}(\vec{r},t)=\rho_e(\vec{r},t)$$
$$\nabla\times\vec{H}(\vec{r},t)=\vec{J}_e(\vec{r},t)+\frac{\partial\vec{D}(\vec{r},t)}{\partial t}\ ,\ \nabla\cdot\vec{B}(\vec{r},t)=\rho_h(\vec{r},t)$$

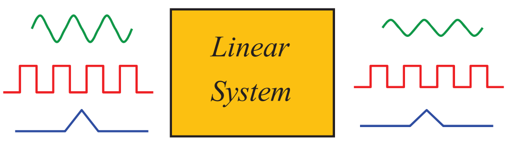
... *time domain Maxwell equations (TDME's)*

Excitation (Input) — *Response (Output)* — *Analysis Method*

Time Harmonic — *Phasor* $(\omega=\omega_0=single\ discrete\ frequency)$

Periodic — *Fourier Series* $(\omega=\omega_n=n^{th}\ discrete\ frequency)$

Aperiodic — *Fourier Transform* $(\omega=continuous\ frequency)$



$$\vec{F}(\vec{r},t)=\mathrm{Re}\{\vec{F}(\vec{r},\omega_0)e^{j\omega_0 t}\}\ \ (\vec{F}=generic\ field:\vec{E},\vec{J}_h,etc.)\ \rightarrow TDME's\underset{\frac{\partial}{\partial t}\rightarrow j\omega_0}{\Rightarrow}$$
$$\nabla\times\vec{E}(\vec{r},\omega_0)=-\vec{J}_h(\vec{r},\omega_0)-j\omega_0\vec{B}(\vec{r},\omega_0)\ ,\ \nabla\cdot\vec{D}(\vec{r},\omega_0)=\rho_e(\vec{r},\omega_0)$$
$$\nabla\times\vec{H}(\vec{r},\omega_0)=\vec{J}_e(\vec{r},\omega_0)+j\omega_0\vec{D}(\vec{r},\omega_0)\ ,\ \nabla\cdot\vec{B}(\vec{r},\omega_0)=\rho_h(\vec{r},\omega_0)$$

$$\vec{F}(\vec{r},t)=\sum_{n=-\infty}^{\infty}\vec{F}_n(\vec{r},\omega_n)e^{j\omega_n t}\rightarrow TDME's\underset{\frac{\partial}{\partial t}\rightarrow\omega_n}{\Rightarrow}$$
$$\nabla\times\vec{E}_n(\vec{r},\omega_n)=-\vec{J}_{hn}(\vec{r},\omega_n)-j\omega_n\vec{B}_n(\vec{r},\omega_n)\ ,\ \nabla\cdot\vec{D}_n(\vec{r},\omega_n)=\rho_{en}(\vec{r},\omega_n)$$
$$\nabla\times\vec{H}_n(\vec{r},\omega_n)=\vec{J}_{en}(\vec{r},\omega_n)+j\omega_n\vec{D}_n(\vec{r},\omega_n)\ ,\ \nabla\cdot\vec{B}_n(\vec{r},\omega_n)=\rho_{hn}(\vec{r},\omega_n)$$

$$\vec{F}(\vec{r},t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\vec{F}(\vec{r},\omega)e^{j\omega t}d\omega\rightarrow TDME's\underset{\frac{\partial}{\partial t}\rightarrow j\omega}{\Rightarrow}$$
$$\nabla\times\vec{E}(\vec{r},\omega)=-\vec{J}_h(\vec{r},\omega)-j\omega\vec{B}(\vec{r},\omega)\ ,\ \nabla\cdot\vec{D}(\vec{r},\omega)=\rho_e(\vec{r},\omega)$$
$$\nabla\times\vec{H}(\vec{r},\omega)=\vec{J}_e(\vec{r},\omega)+j\omega\vec{D}(\vec{r},\omega)\ ,\ \nabla\cdot\vec{B}(\vec{r},\omega)=\rho_h(\vec{r},\omega)$$

The transform domain Maxwell equations all have the same fundamental form!

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Maxwell Equations – Key Take-Aways!

KEY Take-Aways

$$\textit{Technology pushing limits} \Rightarrow$$
$$\textit{Need to understand the various forms of Maxwell}$$
$$\textit{equations and their regimes of use / validity!!}$$

$$\textit{Different mathematical formulations (vector, tensor, Clifford, etc.)}$$
$$\textit{provide varying mathematical and physically insightful advantages.}$$

Maxwell Equations – Homework

For an overview of the tensor formulation of Maxwell's equations, read Chapter 7 of J. Kong, "Electromagnetic Wave Theory, Second Edition," John Wiley, 1990.

Show that $\frac{\partial F_{\alpha\beta}}{\partial x^{\gamma}} + \frac{\partial F_{\beta\gamma}}{\partial x^{\alpha}} + \frac{\partial F_{\gamma\alpha}}{\partial x^{\beta}} = -M_{\alpha\beta\gamma}\ ,\ \frac{\partial G^{\alpha\mu}}{\partial x^{\alpha}} = J^{\mu}$ *are indeed Maxwell's equations.*

For a brief overview of Clifford (i.e., geometric) algebra, read the paper: J. Chappell, et al., "Geometric Algebra for Electrical and Electronic Engineers", *Proceedings of the IEEE*, Vol. 102, No. 9, September 2014.

For an application of general relativity in electrical engineering, read the paper: U. Leonhardt, et al., "General Relativity in Electrical Engineering", *New Journal of Physics*, Vol. 8, 2006, (doi:10.1088/1367-2630/8/10/247).

Show that the continuity equations $\nabla \cdot \vec{J}_e = -\frac{\partial \rho_e}{\partial t}, \nabla \cdot \vec{J}_h = -\frac{\partial \rho_h}{\partial t}$ *can be derived from Maxwell's equations.*

Well Posed EM Model

Maxwell Equations

$$\nabla \cdot \vec{D} = \rho_e$$
$$\nabla \times \vec{H} = \vec{J}_e + \frac{\partial \vec{D}}{\partial t}$$
$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$
$$\nabla \cdot \vec{B} = 0$$

$+$

Constitutive Relations

$$\vec{D} = \varepsilon_0 \vec{E}$$
$$\vec{B} = \mu_0 \vec{H}$$

... for example

$+$

Lorentz Force

$$\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}$$

$+$

Boundary Conditions

$$\hat{n} \times (\vec{E}_2 - \vec{E}_1) = -\vec{J}_{hs}$$
$$\hat{n} \times (\vec{H}_2 - \vec{H}_1) = \vec{J}_{es}$$
$$\hat{n} \cdot (\vec{D}_2 - \vec{D}_1) = \rho_{es}$$
$$\hat{n} \cdot (\vec{B}_2 - \vec{B}_1) = \rho_{hs}$$

$=$

WELL POSED / UNIQUE MATHEMATICAL AND PHYSICAL MODEL

$$\vec{E} = \lim_{q \to 0^+} \frac{\vec{F}}{q}\Big|_{\vec{v}=0}$$ *...a PUSH / PULL Force*

$$\vec{v} \times \vec{B} = \lim_{q \to 0^+} \frac{\vec{F} - q\vec{E}}{q}\Big|_{\vec{v}\neq 0}$$ *...a DEFLECTIVE Force*

Note : $\hat{n}$ = unit normal pointing from region 1 into region 2.

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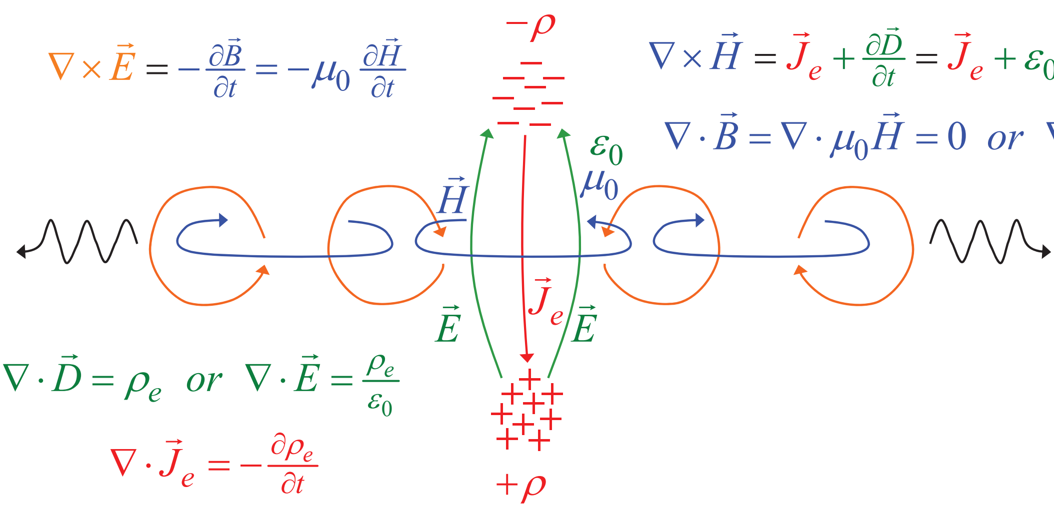
Maxwell Equations in Free Space

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\mu_0 \frac{\partial \vec{H}}{\partial t}$$

$$\nabla \times \vec{H} = \vec{J}_e + \frac{\partial \vec{D}}{\partial t} = \vec{J}_e + \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\nabla \cdot \vec{B} = \nabla \cdot \mu_0 \vec{H} = 0 \quad or \quad \nabla \cdot \vec{H} = 0$$



$$\nabla \cdot \vec{D} = \rho_e \quad or \quad \nabla \cdot \vec{E} = \frac{\rho_e}{\varepsilon_0}$$

$$\nabla \cdot \vec{J}_e = -\frac{\partial \rho_e}{\partial t}$$

Waves in free space 

Waves in complex media 

Need to explore constitutive relations!

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Constitutive Relation Choices

$$\begin{matrix}\vec{D}=\vec{D}(\vec{E},\vec{H}) \\ \vec{B}=\vec{B}(\vec{E},\vec{H})\end{matrix}\;,\quad \begin{matrix}\vec{E}=\vec{E}(\vec{D},\vec{B}) \\ \vec{H}=\vec{H}(\vec{D},\vec{B})\end{matrix}\;,\quad \begin{matrix}\vec{D}=\vec{D}(\vec{E},\vec{B}) \\ \vec{H}=\vec{H}(\vec{E},\vec{B})\end{matrix}$$

Good form when enforcing boundary conditions and determining power flow direction (often the EE standard).

Good form when dealing with plane waves in complex media.

Good form to use when material bodies in motion.

$$\begin{bmatrix}\vec{D}\\ \vec{B}\end{bmatrix}=\underbrace{\begin{bmatrix}\overleftrightarrow{\varepsilon} & \overleftrightarrow{\xi}\\ \overleftrightarrow{\zeta} & \overleftrightarrow{\mu}\end{bmatrix}}_{\overleftrightarrow{C}_{EH}}\cdot\begin{bmatrix}\vec{E}\\ \vec{H}\end{bmatrix}\qquad \begin{bmatrix}\vec{E}\\ \vec{H}\end{bmatrix}=\underbrace{\begin{bmatrix}\overleftrightarrow{\kappa} & \overleftrightarrow{\chi}\\ \overleftrightarrow{\gamma} & \overleftrightarrow{\nu}\end{bmatrix}}_{\overleftrightarrow{C}_{DB}}\cdot\begin{bmatrix}\vec{D}\\ \vec{B}\end{bmatrix}\qquad \begin{bmatrix}c\vec{D}\\ \vec{H}\end{bmatrix}=\underbrace{\begin{bmatrix}\overleftrightarrow{P} & \overleftrightarrow{L}\\ \overleftrightarrow{M} & \overleftrightarrow{Q}\end{bmatrix}}_{\overleftrightarrow{C}_{EB}}\cdot\begin{bmatrix}\vec{E}\\ c\vec{B}\end{bmatrix}$$

In general, $\overleftrightarrow{C}$ can contain differential or integral operators.

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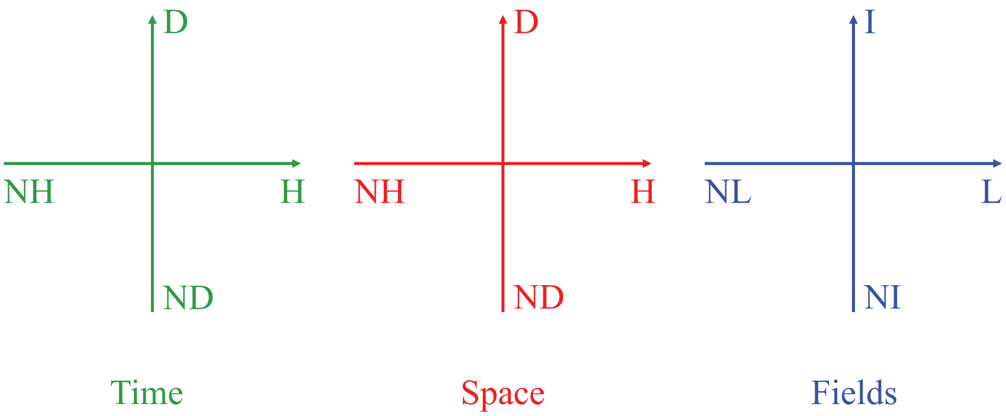
Constitutive Relations - Overview

$$\begin{matrix}\vec{D}[\vec{E}(\vec{r},t),\vec{H}(\vec{r},t)] \\ \vec{B}[\vec{E}(\vec{r},t),\vec{H}(\vec{r},t)]\end{matrix}\;\cdots\;\begin{matrix}\textit{functions of time, space and fields} \\ (\textit{and other factors, e.g., heat, stress})\end{matrix}$$



H/NH=Homogeneous/Not Homogeneous
D/ND=Dispersive/Not Dispersive

L/NL=Linear/Not Linear
I/NI=Isotropic/Not Isotropic

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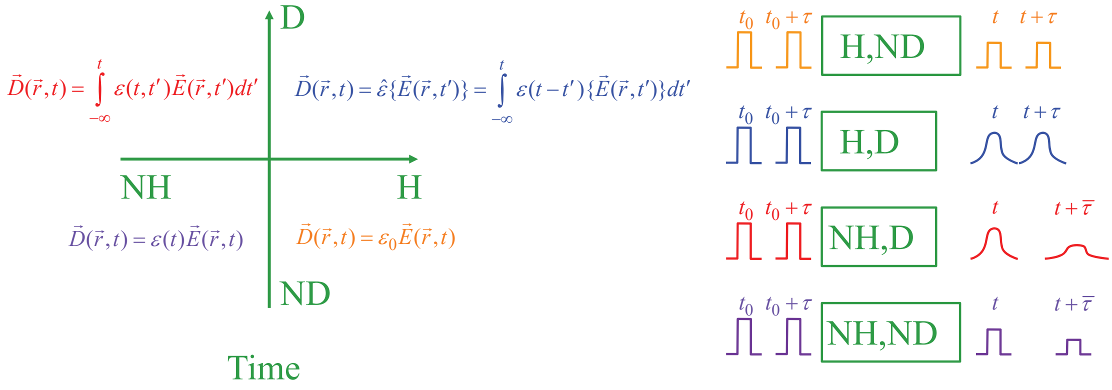
Constitutive Relations – Temporal Definitions and Examples

Temporally Homogeneous – time independent (time invariant).

Temporally Non Homogeneous – time dependent/temporally inhomogeneous (time varying).

Temporally Dispersive/Non-Local – dependent on temporal derivatives/integrals (memory/delay in response).

Temporally Non Dispersive/Local – independent of temporal derivatives/integrals (no memory/instant response).



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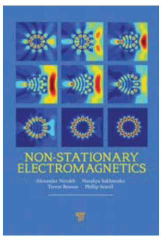
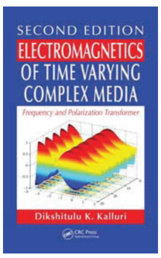
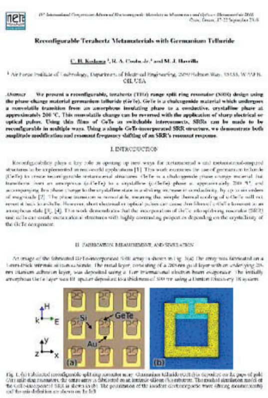
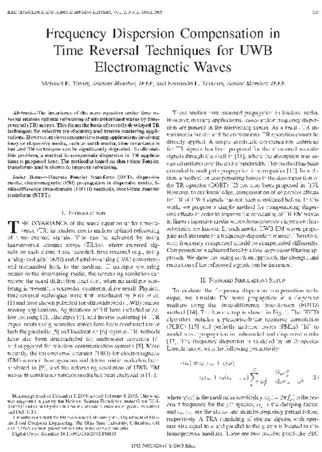
Constitutive Relations – Temporal Definitions and Examples

Applications: Modulation of amplitude/phase/polarization, delay lines, reconfigurable media, dispersion matching or compensation, mode propagation control, resonance control, non-reciprocal media … to name just a few.









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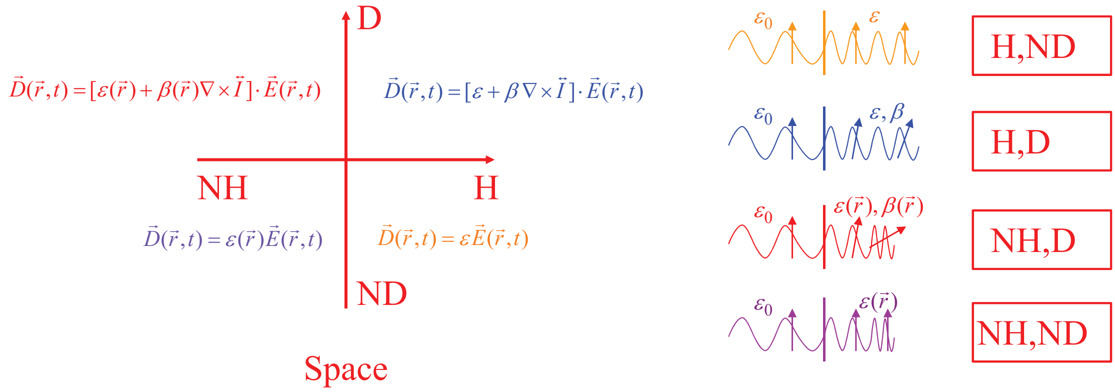
Constitutive Relations – Spatial Definitions and Examples

Spatially Homogeneous – position independent/spatially invariant.

Spatially Non Homogeneous – position dependent/spatially inhomogeneous/varying.

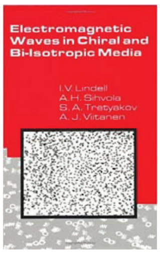
Spatially Dispersive/Non-Local – dependent on spatial derivatives/integrals.

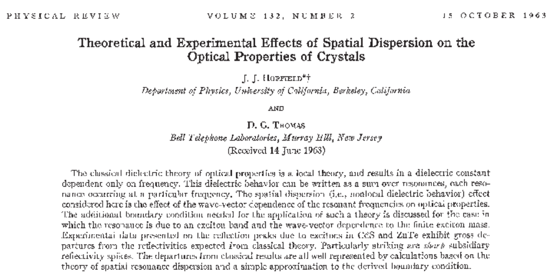
Spatially Non Dispersive/Local – independent of spatial derivatives/integrals.



Constitutive Relations – Spatial Definitions and Examples

Applications: Polarization control, cloaking, radiation enhancement, surface wave control, impedance matching, etc.





PHYSICAL REVIEW VOLUME 132, NUMBER 2 15 OCTOBER 1963

Theoretical and Experimental Effects of Spatial Dispersion on the Optical Properties of Crystals

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AND
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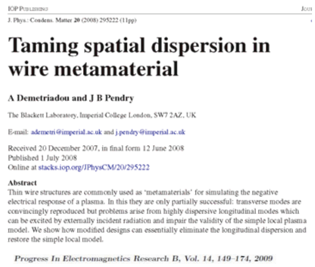
(Received 14 June 1963)

The classical dielectric theory of optical properties is a local theory, and results in a dielectric constant dependent only on frequency. This dielectric behavior can be written as a sum over resonances, each resonance occurring at a particular frequency. The spatial dispersion (i.e., nonlocal dielectric behavior) effect considered here is the effect of the wave-vector dependence of the resonant frequencies on optical properties. The additional boundary condition needed for the application of such a theory is discussed for the case in which the resonance is due to an exciton band and the wave-vector dependence to the finite exciton mass. Experimental data presented on the reflection peaks due to excitons in CdS and ZnTe exhibit gross departures from the reflectivities expected from classical theory. Particularly striking are *sharp* subsidiary reflectivity spikes. The departures from classical results are all well represented by calculations based on the theory of spatial resonance dispersion and a simple approximation to the derived boundary condition.



IOP Publishing
J. Phys.: Condens. Matter **20** (2008) 295222 (11pp)

Taming spatial dispersion in wire metamaterial

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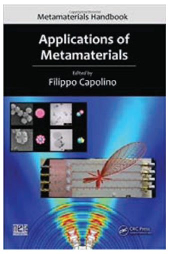
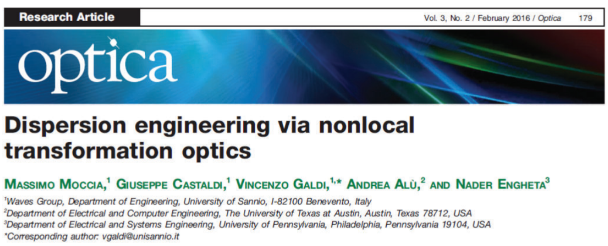
Abstract
Thin wire structures are commonly used as 'metamaterials' for simulating the negative electrical response of a plasma. In this they are only partially successful: transverse modes are convincingly reproduced but problems arise from highly dispersive longitudinal modes which can be excited by externally incident radiation and impair the validity of the simple local plasma model. We show how modified designs can essentially eliminate the longitudinal dispersion and restore the simple local model.





Research Article Vol. 3, No. 2 / February 2016 / *Optica* 179

optica

Dispersion engineering via nonlocal transformation optics

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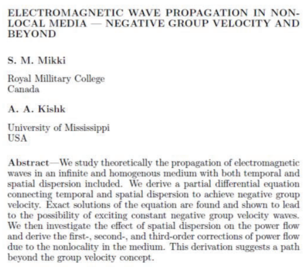

Progress In Electromagnetics Research B, Vol. 14, 149–174, 2009

ELECTROMAGNETIC WAVE PROPAGATION IN NON-LOCAL MEDIA — NEGATIVE GROUP VELOCITY AND BEYOND

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Abstract—We study theoretically the propagation of electromagnetic waves in an infinite and homogenous medium with both temporal and spatial dispersion included. We derive a partial differential equation connecting temporal and spatial dispersion to achieve negative group velocity. Exact solutions of the equation are found and shown to lead to the possibility of exciting constant negative group velocity waves. We then investigate the effect of spatial dispersion on the power flow and derive the first-, second-, and third-order corrections of power flow due to the nonlocality in the medium. This derivation suggests a path beyond the group velocity concept.

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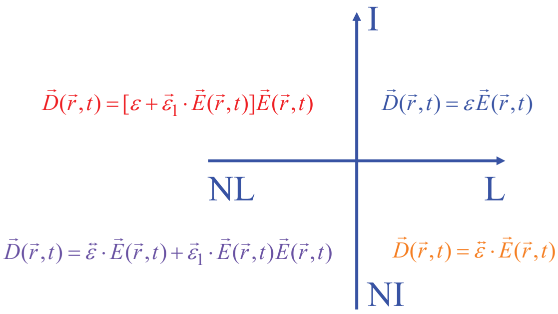
Constitutive Relations – Field Definitions and Examples

Linear – independent of field strength.

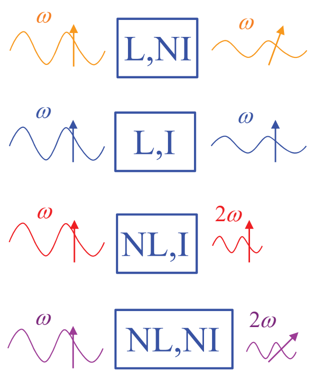
Non Linear – dependent on field strength.

Isotropic – independent of field orientation.

Non Isotropic – dependent on field orientation.

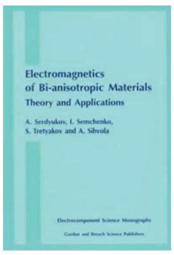


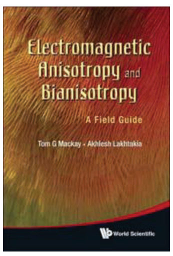
Fields

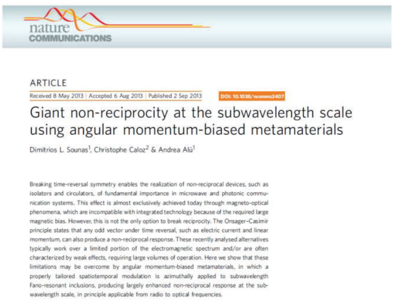


Constitutive Relations – Field Definitions and Examples

Applications: Polarization control, non reciprocal media, cloaking, radiation control, frequency conversion, etc.







ARTICLE

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Giant non-reciprocity at the subwavelength scale using angular momentum-biased metamaterials

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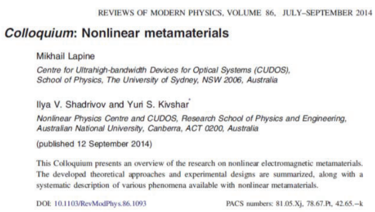
Breaking time-reversal symmetry enables the realization of non-reciprocal devices, such as isolators and circulators, of fundamental importance in microwave and photonic communication systems. This effect is almost exclusively achieved today through magneto-optical phenomena, which are incompatible with integrated technology because of the required large magnetic bias. However, this is not the only option to break reciprocity. The Onsager–Casimir principle states that any odd vector under time reversal, such as electric current and linear momentum, can also produce a non-reciprocal response. These recently analysed alternatives typically work over a limited portion of the electromagnetic spectrum and/or are often characterized by weak effects, requiring large volumes of operation. Here we show that these limitations may be overcome by angular momentum-biased metamaterials, in which a properly tailored spatiotemporal modulation is azimuthally applied to subwavelength Fano-resonant inclusions, producing largely enhanced non-reciprocal response at the subwavelength scale, in principle applicable from radio to optical frequencies.



REVIEWS OF MODERN PHYSICS, VOLUME 86, JULY–SEPTEMBER 2014

Colloquium: Nonlinear metamaterials

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(published 12 September 2014)

This Colloquium presents an overview of the research on nonlinear electromagnetic metamaterials. The developed theoretical approaches and experimental designs are summarized, along with a systematic description of various phenomena available with nonlinear metamaterials.

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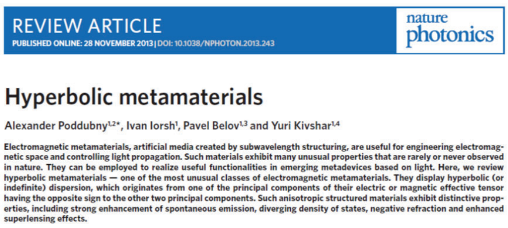


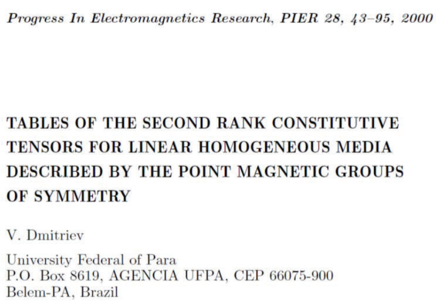
REVIEW ARTICLE nature photonics

PUBLISHED ONLINE: 28 NOVEMBER 2013 | DOI: 10.1038/NPHOTON.2013.243

Hyperbolic metamaterials

Alexander Poddubny[1,2]*, Ivan Iorsh[1], Pavel Belov[1,3] and Yuri Kivshar[1,4]

Electromagnetic metamaterials, artificial media created by subwavelength structuring, are useful for engineering electromagnetic space and controlling light propagation. Such materials exhibit many unusual properties that are rarely or never observed in nature. They can be employed to realize useful functionalities in emerging metadevices based on light. Here, we review hyperbolic metamaterials — one of the most unusual classes of electromagnetic metamaterials. They display hyperbolic (or indefinite) dispersion, which originates from one of the principal components of their electric or magnetic effective tensor having the opposite sign to the other two principal components. Such anisotropic structured materials exhibit distinctive properties, including strong enhancement of spontaneous emission, diverging density of states, negative refraction and enhanced superlensing effects.



Progress In Electromagnetics Research, PIER 28, 43–95, 2000

TABLES OF THE SECOND RANK CONSTITUTIVE TENSORS FOR LINEAR HOMOGENEOUS MEDIA DESCRIBED BY THE POINT MAGNETIC GROUPS OF SYMMETRY

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Constitutive Relations – General Linear Media (Parameters Independent of Field Strength)

LINEAR, BIANISOTROPIC, SPATIALLY VARYING, SPATIALLY DISPERSIVE, TEMPORALLY VARYING, TEMPORALLY DISPERSIVE

Spatially Dispersive

Temporally Dispersive (upper limit "t" for causality)

[F]=[C/V] [A]=[C/s]

[C/m²] [(F/m)/(m³s)] [V/m] [s] [m³] [(s/m)/(m³s)] [A/m] [s] [m³]

$$\vec{D}(\vec{r},t)=\int_V\int_{-\infty}^{t}\vec{\vec{\varepsilon}}(\vec{r},\vec{r}',t,t')\cdot\vec{E}(\vec{r}',t')dt'dV'+\int_V\int_{-\infty}^{t}\vec{\vec{\xi}}(\vec{r},\vec{r}',t,t')\cdot\vec{H}(\vec{r}',t')dt'dV'$$

Bianisotropic $(\vec{\vec{\varepsilon}},\vec{\vec{\mu}},\vec{\vec{\xi}},\vec{\vec{\zeta}})$

[Wb/m²] [(s/m)/(m³s)] [V/m] [s] [m³] [H/(m⁴s)] [A/m] [s] [m³]

$$\vec{B}(\vec{r},t)=\int_V\int_{-\infty}^{t}\vec{\vec{\zeta}}(\vec{r},\vec{r}',t,t')\cdot\vec{E}(\vec{r}',t')dt'dV'+\int_V\int_{-\infty}^{t}\vec{\vec{\mu}}(\vec{r},\vec{r}',t,t')\cdot\vec{H}(\vec{r}',t')dt'dV'$$

[Wb]=[Vs] [H]=[Wb/A]

Spatially Inhomogeneous/ Spatially Varying

Linear since not a function of the fields

Temporally Inhomogeneous/ Time Varying

$$\vec{\vec{\varepsilon}}(\vec{r},\vec{r}',t,t'),\vec{\vec{\mu}}(\vec{r},\vec{r}',t,t'),\vec{\vec{\xi}}(\vec{r},\vec{r}',t,t'),\vec{\vec{\zeta}}(\vec{r},\vec{r}',t,t')\ldots$$

Constitutive Parameters (Dyadic Green's Functions!)

TRANSFORM DOMAIN ANALYSIS NOT TOO HELPFUL FOR THIS GENERAL CASE

Note: Spatial integration is carried over the region in which $|\vec{r}-\vec{r}'|\le c(t-t')$.

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Constitutive Relations – General Linear Media (Examples)

LINEAR, BIANISOTROPIC, SPATIALLY INVARIANT, SPATIALLY DISPERSIVE, TEMPORALLY INVARIANT, TEMPORALLY DISPERSIVE

$$\vec{D}(\vec{r},t)=\int_V\int_{-\infty}^{t}\vec{\vec{\varepsilon}}(\vec{r}-\vec{r}',t-t')\cdot\vec{E}(\vec{r}',t')dt'dV'+\int_V\int_{-\infty}^{t}\vec{\vec{\xi}}(\vec{r}-\vec{r}',t-t')\cdot\vec{H}(\vec{r}',t')dt'dV'$$

$$\vec{B}(\vec{r},t)=\int_V\int_{-\infty}^{t}\vec{\vec{\zeta}}(\vec{r}-\vec{r}',t-t')\cdot\vec{E}(\vec{r}',t')dt'dV'+\int_V\int_{-\infty}^{t}\vec{\vec{\mu}}(\vec{r}-\vec{r}',t-t')\cdot\vec{H}(\vec{r}',t')dt'dV'$$

... time domain relations

$$\vec{D}(\vec{k},\omega)=\vec{\vec{\varepsilon}}(\vec{k},\omega)\cdot\vec{E}(\vec{k},\omega)+\vec{\vec{\xi}}(\vec{k},\omega)\cdot\vec{H}(\vec{k},\omega)$$

$$\vec{B}(\vec{k},\omega)=\vec{\vec{\zeta}}(\vec{k},\omega)\cdot\vec{E}(\vec{k},\omega)+\vec{\vec{\mu}}(\vec{k},\omega)\cdot\vec{H}(\vec{k},\omega)$$

...transform domain relations

TRANSFORM DOMAIN ANALYSIS MOST AMENABLE FOR THIS SPECIAL CASE

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Constitutive Relations – General Linear Media (Examples)

LINEAR, BIANISOTROPIC, SPATIALLY INVARIANT, SPATIALLY LOCAL, TEMPORALLY INVARIANT, TEMPORALLY LOCAL

$$\vec{D}(\vec{r},t)=\int_V\int_{-\infty}^{t}\vec{\vec{\varepsilon}}\,\delta(\vec{r}-\vec{r}\,')\delta(t-t')\cdot\vec{E}(\vec{r}\,',t')dt'dV'+\int_V\int_{-\infty}^{t}\vec{\vec{\xi}}\,\delta(\vec{r}-\vec{r}\,')\delta(t-t')\cdot\vec{H}(\vec{r}\,',t')dt'dV'$$

$$\vec{B}(\vec{r},t)=\int_V\int_{-\infty}^{t}\vec{\vec{\zeta}}\,\delta(\vec{r}-\vec{r}\,')\delta(t-t')\cdot\vec{E}(\vec{r}\,',t')dt'dV'+\int_V\int_{-\infty}^{t}\vec{\vec{\mu}}\,\delta(\vec{r}-\vec{r}\,')\delta(t-t')\cdot\vec{H}(\vec{r}\,',t')dt'dV'$$

$$\Rightarrow$$

$$\left.\begin{aligned}\vec{D}(\vec{r},t)&=\vec{\vec{\varepsilon}}\cdot\vec{E}(\vec{r},t)+\vec{\vec{\xi}}\cdot\vec{H}(\vec{r},t)\\ \vec{B}(\vec{r},t)&=\vec{\vec{\zeta}}\cdot\vec{E}(\vec{r},t)+\vec{\vec{\mu}}\cdot\vec{H}(\vec{r},t)\end{aligned}\right.\ldots\ \textit{time domain relations (not physically realistic)}$$

$$\left.\begin{aligned}\vec{D}(\vec{r},\omega)&=\vec{\vec{\varepsilon}}\cdot\vec{E}(\vec{r},\omega)+\vec{\vec{\xi}}\cdot\vec{H}(\vec{r},\omega)\\ \vec{B}(\vec{r},\omega)&=\vec{\vec{\zeta}}\cdot\vec{E}(\vec{r},\omega)+\vec{\vec{\mu}}\cdot\vec{H}(\vec{r},\omega)\end{aligned}\right.\ldots\textit{transform domain relations}$$

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Constitutive Relations – SIMPLE Media

LINEAR, ISOTROPIC, SPATIALLY INVARIANT, SPATIALLY LOCAL, TEMPORALLY INVARIANT, TEMPORALLY DISPERSIVE

$$\vec{D}(\vec{r},t)=\int_V\int_{-\infty}^{t}\varepsilon(t-t')\delta(\vec{r}-\vec{r}\,')\vec{\vec{I}}\cdot\vec{E}(\vec{r}\,',t')dt'dV'$$

$$\vec{B}(\vec{r},t)=\int_V\int_{-\infty}^{t}\mu(t-t')\delta(\vec{r}-\vec{r}\,')\vec{\vec{I}}\cdot\vec{H}(\vec{r}\,',t')dt'dV'$$

$$\vec{\vec{I}}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$\Rightarrow$$

$$\left.\begin{aligned}\vec{D}(\vec{r},t)&=\int_{-\infty}^{t}\varepsilon(t-t')\vec{\vec{I}}\cdot\vec{E}(\vec{r},t')dt'\\ \vec{B}(\vec{r},t)&=\int_{-\infty}^{t}\mu(t-t')\vec{\vec{I}}\cdot\vec{H}(\vec{r},t')dt'\end{aligned}\right.\ldots\ \textit{time domain relations (SIMPLE media)}$$

$$\left.\begin{aligned}\vec{D}(\vec{r},\omega)&=\varepsilon(\omega)\vec{\vec{I}}\cdot\vec{E}(\vec{r},\omega)=\varepsilon(\omega)\vec{E}(\vec{r},\omega)\\ \vec{B}(\vec{r},\omega)&=\mu(\omega)\vec{\vec{I}}\cdot\vec{H}(\vec{r},\omega)=\mu(\omega)\vec{H}(\vec{r},\omega)\end{aligned}\right.\ldots\ \textit{transform domain relations (SIMPLE media)}$$

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Constitutive Relations – SIMPLE Media (Taylor Series Example)

$$\vec{D}(\vec{r},t)=\int_{-\infty}^{t}\varepsilon(t-t')\vec{E}(\vec{r},t')dt'=\int_{-\infty}^{t}\varepsilon(t-t')\sum_{n=0}^{\infty}\frac{1}{n!}\frac{\partial^n\vec{E}(\vec{r},t)}{\partial t^n}(t-t')^n dt'=\sum_{n=0}^{\infty}\frac{\partial^n\vec{E}(\vec{r},t)}{\partial t^n}\overbrace{\frac{1}{n!}\int_{-\infty}^{t}\varepsilon(t-t')(t-t')^n dt'}^{\varepsilon_n}$$

$$\Rightarrow\ \vec{D}(\vec{r},t)=\sum_{n=0}^{\infty}\varepsilon_n\frac{\partial^n\vec{E}(\vec{r},t)}{\partial t^n}=\varepsilon_0\vec{E}(\vec{r},t)+\varepsilon_1\frac{\partial\vec{E}(\vec{r},t)}{\partial t}+\varepsilon_2\frac{\partial^2\vec{E}(\vec{r},t)}{\partial t^2}+\varepsilon_3\frac{\partial^3\vec{E}(\vec{r},t)}{\partial t^3}+\cdots$$

$$\begin{aligned}\vec{D}(\vec{r},\omega)&=\varepsilon_0\vec{E}(\vec{r},\omega)+\varepsilon_1 j\omega\vec{E}(\vec{r},\omega)-\varepsilon_2\omega^2\vec{E}(\vec{r},\omega)-\varepsilon_3 j\omega^3\vec{E}(\vec{r},\omega)+\cdots\\&=[\underbrace{(\varepsilon_0-\varepsilon_2\omega^2+\cdots)}_{even\ function\ of\ \omega}+j\underbrace{(\varepsilon_1\omega-\varepsilon_3\omega^3+\cdots)}_{odd\ function\ of\ \omega}]\vec{E}(\vec{r},\omega)\\&=[\varepsilon_{re}(\omega)+j\varepsilon_{im}(\omega)]\vec{E}(\vec{r},\omega)\ldots a\ common\ result*\end{aligned}$$

* R. Harrington, Time-Harmonic Electromagnetic Fields, IEEE Press 2001 (pg. 6,18).

For a discussion supporting the integral form of the constitutive relations, see A. Lakhtakia and W. Weiglhofer, "Are Field Derivatives Needed in Linear Constitutive Relations", *International Journal of Infrared and Millimeter Waves*, Vol. 19, No. 8, 1998.

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Constitutive Relations – Nonlinear Media (Example)

NONLINEAR, ANISOTROPIC, SPATIALLY VARYING, SPATIALLY NONLOCAL, TEMPORALLY VARYING, TEMPORALLY NONLOCAL

$$D_\alpha(\vec{r},t)=\overbrace{\int_{V'}\int_{t'=-\infty}^{t}\varepsilon_{\alpha\beta}(\vec{r},\vec{r}',t,t')E_\beta(\vec{r}',t')dt'dV'}^{LINEAR\ TERM}$$
$$\underbrace{+\int_{V''}\int_{t''=-\infty}^{t}\int_{V'}\int_{t'=-\infty}^{t}\varepsilon_{\alpha\beta\gamma}(\vec{r},\vec{r}',\vec{r}'',t,t',t'')E_\beta(\vec{r}',t')E_\gamma(\vec{r}'',t'')dt'dV'dt''dV''+\cdots}_{NONLINEAR\ TERMS*}$$

$$or\ \ \vec{D}(\vec{r},t)=\overbrace{\int_{V'}\int_{t'=-\infty}^{t}\overline{\overline{\varepsilon}}(\vec{r},\vec{r}',t,t')\cdot\vec{E}(\vec{r}',t')dt'dV'}^{LINEAR\ TERM}$$
$$\underbrace{+\int_{V''}\int_{t''=-\infty}^{t}\int_{V'}\int_{t'=-\infty}^{t}[\overline{\overline{\overline{\varepsilon}}}(\vec{r},\vec{r}',\vec{r}'',t,t',t'')\cdot\vec{E}(\vec{r}'',t'')]\cdot\vec{E}(\vec{r}',t')dt'dV'dt''dV''+\cdots}_{NONLINEAR\ TERMS*}$$

* Y. Il'inskii and L. Keldysh, Electromagnetic Response of Material Media, Plenum, 1994.

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Constitutive Relations – Outside the SIMPLE Media Box

Lots of research going on outside the SIMPLE media box!

COMPLEX MEDIA

Nonlinear, Bianisotropic, Spatially Inhomogeneous, Spatially Nonlocal, Time-Varying, Temporally Dispersive (plus other effects such as heat dependent, stress dependent, etc.)

SIMPLE MEDIA
Linear, Isotropic, Spatially Homogeneous
Spatially Local, Time Invariant, Temporally Dispersive

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Constitutive Relations – Media Considered In This Course

LINEAR, BIANISOTROPIC, SPATIALLY INVARIANT, SPATIALLY LOCAL, TEMPORALLY INVARIANT, TEMPORALLY DISPERSIVE

$$\vec{D}(\vec{r},t) = \int_{-\infty}^{t} \overset{\leftrightarrow}{\varepsilon}(t-t')\cdot\vec{E}(\vec{r},t')dt' + \int_{-\infty}^{t} \overset{\leftrightarrow}{\xi}(t-t')\cdot\vec{H}(\vec{r},t')dt'$$

$$\vec{B}(\vec{r},t) = \int_{-\infty}^{t} \overset{\leftrightarrow}{\zeta}(t-t')\cdot\vec{E}(\vec{r},t')dt' + \int_{-\infty}^{t} \overset{\leftrightarrow}{\mu}(t-t')\cdot\vec{H}(\vec{r},t')dt'$$

... time domain relations

$$\vec{D}(\vec{r},\omega) = \overset{\leftrightarrow}{\varepsilon}(\omega)\cdot\vec{E}(\vec{r},\omega) + \overset{\leftrightarrow}{\xi}(\omega)\cdot\vec{H}(\vec{r},\omega)$$

$$\vec{B}(\vec{r},\omega) = \overset{\leftrightarrow}{\zeta}(\omega)\cdot\vec{E}(\vec{r},\omega) + \overset{\leftrightarrow}{\mu}(\omega)\cdot\vec{H}(\vec{r},\omega)$$

...transform domain relations

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Motivation – Complex Media Gives More Control Over EM Field

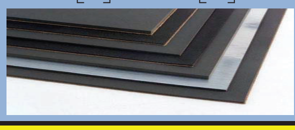
Complex Media

BIISOTROPIC (4)

$$\begin{bmatrix}\vec{D}\\ \vec{B}\end{bmatrix}=\begin{bmatrix}\varepsilon & \xi\\ \zeta & \mu\end{bmatrix}\begin{bmatrix}\vec{E}\\ \vec{H}\end{bmatrix}$$

ISOTROPIC (2)

$$\begin{bmatrix}\vec{D}\\ \vec{B}\end{bmatrix}=\begin{bmatrix}\varepsilon & 0\\ 0 & \mu\end{bmatrix}\begin{bmatrix}\vec{E}\\ \vec{H}\end{bmatrix}$$

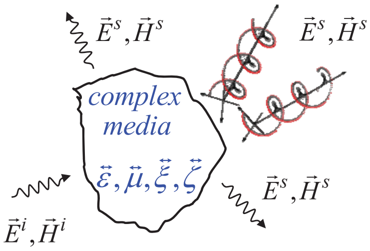


BIANISOTROPIC (36)

$$\begin{bmatrix}\vec{D}\\ \vec{B}\end{bmatrix}=\begin{bmatrix}\vec{\vec{\varepsilon}} & \vec{\vec{\xi}}\\ \vec{\vec{\zeta}} & \vec{\vec{\mu}}\end{bmatrix}\cdot\begin{bmatrix}\vec{E}\\ \vec{H}\end{bmatrix}$$

ANISOTROPIC (18)

$$\begin{bmatrix}\vec{D}\\ \vec{B}\end{bmatrix}=\begin{bmatrix}\vec{\vec{\varepsilon}} & \vec{\vec{0}}\\ \vec{\vec{0}} & \vec{\vec{\mu}}\end{bmatrix}\cdot\begin{bmatrix}\vec{E}\\ \vec{H}\end{bmatrix}$$



$$\vec{\vec{\varepsilon}}=\begin{bmatrix}\varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz}\\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz}\\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz}\end{bmatrix},\quad \vec{\vec{\mu}}=\begin{bmatrix}\mu_{xx} & \mu_{xy} & \mu_{xz}\\ \mu_{yx} & \mu_{yy} & \mu_{yz}\\ \mu_{zx} & \mu_{zy} & \mu_{zz}\end{bmatrix}$$

$$\vec{\vec{\xi}}=\begin{bmatrix}\xi_{xx} & \xi_{xy} & \xi_{xz}\\ \xi_{yx} & \xi_{yy} & \xi_{yz}\\ \xi_{zx} & \xi_{zy} & \xi_{zz}\end{bmatrix},\quad \vec{\vec{\zeta}}=\begin{bmatrix}\zeta_{xx} & \zeta_{xy} & \zeta_{xz}\\ \zeta_{yx} & \zeta_{yy} & \zeta_{yz}\\ \zeta_{zx} & \zeta_{zy} & \zeta_{zz}\end{bmatrix}$$

$\vec{\vec{\varepsilon}},\vec{\vec{\mu}},\vec{\vec{\xi}},\vec{\vec{\zeta}} \Rightarrow$ *more control* 😊

$$\nabla\times\vec{E}=-\vec{J}_h-j\omega\vec{B}$$
$$\nabla\times\vec{H}=\vec{J}_e+j\omega\vec{D}$$

Symmetry plays a critical role – as we will see!

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Constitutive Relations – Key Take-Aways!

KEY Take-Aways

Constitutive relations are critical for a well-posed mathematical model.

Materials can be categorized according to how they respond temporally and spatially under field excitation (field = electric, magnetic, temperature, etc.).

Linear, bianisotropic, spatially/temporally invariant, spatially/temporally nonlocal media is most general class amenable to Fourier transformation.

Current research exploring new ways to control EM fields via complex media!

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Constitutive Relations – Homework

Determine the constitutive relations for a linear, bianisotropic, spatially invariant, spatially local, temporally varying, temporally nonlocal media.

Determine the constitutive relations for a linear, bi-isotropic, spatially invariant, spatially local, temporally invariant, temporally nonlocal media.

Determine the constitutive relations for a linear, anisotropic, spatially invariant, spatially nonlocal, temporally varying, temporally nonlocal media.

Determine the constitutive relations for a linear, isotropic, spatially varying, spatially nonlocal, temporally invariant, temporally nonlocal media.

Show the relationship between $\overleftrightarrow{C}_{EH}$ *and* $\overleftrightarrow{C}_{EB}$ *is* $\overleftrightarrow{C}_{EH} = \frac{1}{c}\begin{bmatrix} \overleftrightarrow{P} - \overleftrightarrow{L}\cdot\overleftrightarrow{Q}^{-1}\cdot\overleftrightarrow{M} & \overleftrightarrow{L}\cdot\overleftrightarrow{Q}^{-1} \\ -\overleftrightarrow{Q}^{-1}\cdot\overleftrightarrow{M} & \overleftrightarrow{Q}^{-1} \end{bmatrix}$

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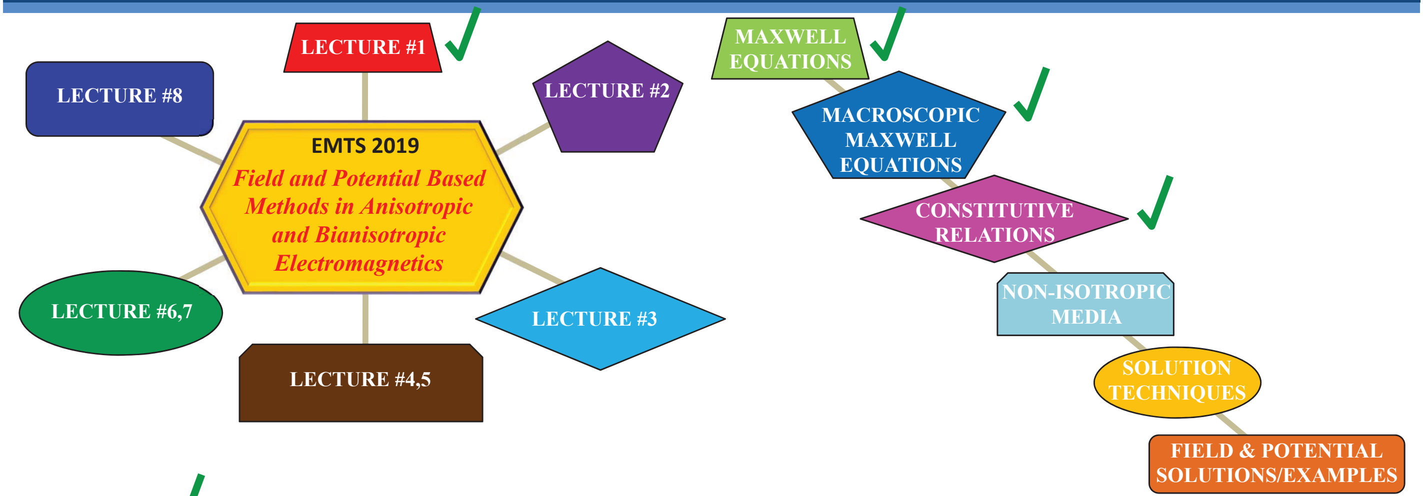
Overview – Lectures/Big Picture



LECTURE #1: ✓ Review various forms and regimes of validity of Maxwell's equations and constitutive relations.
LECTURE #2: Discuss factors that influence anisotropy and bianisotropy.
LECTURE #3: Discuss field and potential based methods for solving electromagnetic problems.
LECTURE #4,5: Provide several examples of field based solutions in complex media within and exterior to source regions.
LECTURE #6,7: Provide several examples of potential based solutions in complex media within and exterior to source regions.
LECTURE #8: Summary, conclusions and future research.

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2019 International Symposium on Electromagnetic Theory

LECTURE #2
Factors that Influence Material Tensor Form

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Factors that Influence Material Tensor Form – Overview

Relative motion.

Weak spatial nonlocality.

Symmetry – an inherent property of naturally occurring media or symmetry that is infused into artificially designed materials.

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Factors that Influence Material Tensor Form – Relative Motion

$$\begin{bmatrix} c\vec{D} \\ \vec{H} \end{bmatrix} = \begin{bmatrix} \overleftrightarrow{P} & \overleftrightarrow{L} \\ \overleftrightarrow{M} & \overleftrightarrow{Q} \end{bmatrix} \cdot \begin{bmatrix} \vec{E} \\ c\vec{B} \end{bmatrix} = \overleftrightarrow{C} \cdot \begin{bmatrix} \vec{E} \\ c\vec{B} \end{bmatrix} = \ldots \quad \textit{constitutive relations ("rest" frame)}$$

$$\begin{bmatrix} c\vec{D}' \\ \vec{H}' \end{bmatrix} = \overleftrightarrow{L}_6 \cdot \overleftrightarrow{C} \cdot \overleftrightarrow{L}_6^{-1} \cdot \begin{bmatrix} \vec{E}' \\ c\vec{B}' \end{bmatrix} = \overleftrightarrow{C}' \cdot \begin{bmatrix} \vec{E}' \\ c\vec{B}' \end{bmatrix} \ldots \quad \textit{constitutive relations in relative moving frame (Lorentz covariant).}$$

$$\overleftrightarrow{L}_6 = \gamma \begin{bmatrix} \overleftrightarrow{\alpha}^{-1} & \overleftrightarrow{\beta} \\ -\overleftrightarrow{\beta} & \overleftrightarrow{\alpha}^{-1} \end{bmatrix}, \ \overleftrightarrow{\alpha}^{-1} = \overleftrightarrow{I} + (\tfrac{1}{\gamma} - 1)\frac{\vec{\beta}\vec{\beta}}{\beta^2}, \ \overleftrightarrow{\beta} = \vec{\beta} \times \overleftrightarrow{I}, \ \vec{\beta} = \vec{v}/c, \ \gamma = (1-\beta^2)^{-\frac{1}{2}}$$

$$\overleftrightarrow{C}' = \overleftrightarrow{L}_6 \cdot \overleftrightarrow{C} \cdot \overleftrightarrow{L}_6^{-1} \ \textit{ or } \ \overleftrightarrow{C} = \overleftrightarrow{L}_6^{-1} \cdot \overleftrightarrow{C}' \cdot \overleftrightarrow{L}_6 \quad \ldots \textit{Note}: \overleftrightarrow{C}' = \overleftrightarrow{C} \ldots \textit{if } \vec{v} = 0.$$

Jin Au Kong, "Electromagnetic Wave Theory, Second Edition," John Wiley, Chapter 7, pp. 585-651, 1990.

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Factors that Influence Material Tensor Form – Relative Motion Example

$$\textit{Assume} \ \begin{matrix} \vec{D}' = \varepsilon' \vec{E}' \\ \vec{B}' = \mu' \vec{H}' \end{matrix} \Rightarrow \begin{bmatrix} c\vec{D}' \\ \vec{H}' \end{bmatrix} = \begin{bmatrix} c\varepsilon' \overleftrightarrow{I} & \overleftrightarrow{0} \\ \overleftrightarrow{0} & \frac{1}{c\mu'} \overleftrightarrow{I} \end{bmatrix} \begin{bmatrix} \vec{E}' \\ c\vec{B}' \end{bmatrix} = \overleftrightarrow{C}' \begin{bmatrix} \vec{E}' \\ c\vec{B}' \end{bmatrix} \ldots \quad \textit{media isotropic in moving frame}$$

$$\overleftrightarrow{C} = \overleftrightarrow{L}_6^{-1} \cdot \overleftrightarrow{C}' \cdot \overleftrightarrow{L}_6 = ? \ (\textit{what material properties does "rest" frame observer see?})$$

$$\overleftrightarrow{C} = \frac{\gamma^2}{c\mu'} \begin{bmatrix} (c^2\varepsilon'\mu' - \beta^2)\overleftrightarrow{I} - (c^2\varepsilon'\mu' - 1)\vec{\beta}\vec{\beta} & (c^2\varepsilon'\mu' - 1)\overleftrightarrow{\beta} \\ (c^2\varepsilon'\mu' - 1)\overleftrightarrow{\beta} & (1 - c^2\varepsilon'\mu'\beta^2)\overleftrightarrow{I} + (c^2\varepsilon'\mu' - 1)\vec{\beta}\vec{\beta} \end{bmatrix} \ldots \quad \underbrace{\textit{bianisotropic in rest frame!}}_{\textit{Is this physically reasonable?}}$$

$$\overleftrightarrow{C} = \frac{1}{c\mu'} \begin{bmatrix} c^2\varepsilon'\mu' \overleftrightarrow{I} & \overleftrightarrow{0} \\ \overleftrightarrow{0} & \overleftrightarrow{I} \end{bmatrix} \ldots \textit{if } \vec{v} = 0.$$

Jin Au Kong, "Electromagnetic Wave Theory, Second Edition," John Wiley, Chapter 7, pp. 585-651, 1990.

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Factors that Influence Material Tensor Form – Weak Spatial Dispersion (Example)

LINEAR, ISOTROPIC, SPATIALLY INVARIANT, SPATIALLY DISPERSIVE, TEMPORALLY INVARIANT, TEMPORALLY DISPERSIVE, NONMAGNETIC

$$\vec{D}(\vec{r},t) = \int_V \int_{-\infty}^{t} \varepsilon(\vec{r}-\vec{r}\,',t-t')\vec{E}(\vec{r}\,',t')dt'dV' \qquad \vec{B}(\vec{r},t) = \mu_0 \vec{H}(\vec{r},t) \;\; or \;\; \vec{H}(\vec{r},t) = \frac{\vec{B}(\vec{r},t)}{\mu_0} \quad \ldots \vec{r},t \; domain$$

$$\Rightarrow$$

$$\vec{D}(\vec{r},\omega) = \int_V \varepsilon(\vec{r}-\vec{r}\,',\omega)\vec{E}(\vec{r}\,',\omega)dV' \qquad \vec{H}(\vec{r},\omega) = \frac{\vec{B}(\vec{r},\omega)}{\mu_0} \quad \ldots \vec{r},\omega \; domain$$

$$\nabla\times\vec{E}(\vec{r},t) = -\frac{\partial\vec{B}(\vec{r},t)}{\partial t} \qquad \nabla\times\vec{H}(\vec{r},t) = \vec{J}_e(\vec{r},t) + \frac{\partial\vec{D}(\vec{r},t)}{\partial t}$$

$$\Rightarrow \quad \nabla\times\vec{E}(\vec{r},\omega) = -j\omega\vec{B}(\vec{r},\omega) \qquad \nabla\times\vec{H}(\vec{r},\omega) = \vec{J}_e(\vec{r},\omega) + j\omega\vec{D}(\vec{r},\omega)$$

A. De Baas, S. Tretyakov, et al., "Nanostructured Metamaterials," European Commission, 2010.

Factors that Influence Material Tensor Form – Weak Spatial Dispersion (Example)

$$\vec{D}(\vec{r},\omega) = \int_V \varepsilon(\vec{r}-\vec{r}\,',\omega)\cdot\vec{E}(\vec{r}\,',\omega)dV' = \varepsilon\vec{E} + \alpha\nabla\times\vec{E} + \beta\nabla\nabla\cdot\vec{E} + \gamma\nabla\times\nabla\times\vec{E} + \cdots \quad (using\ Taylor\ expansion)$$

$$\vec{D}(\vec{r},\omega) \approx \varepsilon\vec{E} + \alpha\nabla\times\vec{E} \ldots if\ spatial\ dispersion\ is\ weak.$$

$$Note: \overbrace{\begin{matrix}\vec{D}' = \vec{D} + \nabla\times\vec{Q} \\ \vec{H}' = \vec{H} + j\omega\vec{Q}\end{matrix}}^{an\ invariant\ transformation} \rightarrow \nabla\times\vec{H} = \vec{J}_e + j\omega\vec{D} \Rightarrow \nabla\times\vec{H}' - \cancel{j\omega\nabla\times\vec{Q}} = \vec{J}_e + j\omega\vec{D}' - \cancel{j\omega\nabla\times\vec{Q}}$$

$$Let\ \vec{Q} = -\frac{\alpha}{2}\vec{E} \Rightarrow \vec{D}' = \vec{D} + \nabla\times\vec{Q} = \varepsilon\vec{E} + \alpha\overbrace{\nabla\times\vec{E}}^{-j\omega\vec{B}} + \overbrace{\nabla\times(-\frac{\alpha}{2}\vec{E})}^{-\frac{\alpha}{2}(-j\omega\vec{B})} = \varepsilon\vec{E} - j\omega\frac{\alpha}{2}\vec{B} \quad \ldots bi-isotropic\ spatially\ local!$$

$$\vec{H}' = \vec{H} + j\omega\vec{Q} = \frac{\vec{B}}{\mu_0} - j\omega\frac{\alpha}{2}\vec{E}$$

LINEAR, BI-ISOTROPIC, SPATIALLY INVARIANT, SPATIALLY LOCAL, TEMPORALLY INVARIANT, TEMPORALLY DISPERSIVE, NONMAGNETIC

Factors that Influence Material Tensor Form – Symmetry

NO SPIN (PAIRED SPIN)

TYPE	DIPOLE ALIGNMENT	EXAMPLES
dielectric	$\vec{E}=0, \vec{P}=0$, $\vec{P}_i=0$; $\vec{E}\neq 0, \vec{P}\neq 0$	Teflon, rexolite, most materials
diaelectric	$\vec{E}=0, \vec{P}=0$, $\vec{P}_i=0$; $\vec{E}\neq 0, \vec{P}\neq 0$	???
paraelectric	$\vec{E}=0, \vec{P}=0$, $\vec{P}_i\neq 0$; $\vec{E}\neq 0, \vec{P}\neq 0$	SiO_2, Al_2O_3
ferroelectric	$\vec{P}_i\neq 0, \vec{P}\neq 0$	Barium titanate, Rochelle salt
antiferroelectric	$\vec{P}_i\neq 0, \vec{P}=0$	$PbZrO_3$, Cesium niobate
ferrielectric	$\vec{P}_i\neq 0, \vec{P}\neq 0$	$DyMn_2O_5$

Electric Dipole

SPIN (UNPAIRED SPIN)

TYPE	SPIN ALIGNMENT	EXAMPLES
dimagnetic	$\vec{H}=0, \vec{M}=0$, $\vec{M}_i=0$; $\vec{H}\neq 0, \vec{M}\neq 0$	???
diamagnetic	$\vec{H}=0, \vec{M}=0$, $\vec{M}_i=0$; $\vec{H}\neq 0, \vec{M}\neq 0$	Copper, silver, gold, water, just about everything
paramagnetic	$\vec{H}=0, \vec{M}=0$, $\vec{M}_i\neq 0$; $\vec{H}\neq 0, \vec{M}\neq 0$	Oxygen, sodium calcium
ferromagnetic	$\vec{M}_i\neq 0, \vec{M}\neq 0$	Iron, cobalt, nickel
antiferromagnetic	$\vec{M}_i\neq 0, \vec{M}=0$	Chromium, FeMn, NiO
ferrimagnetic	$\vec{M}_i\neq 0, \vec{M}\neq 0$	Magnetite, iron garnet

Magnetic Dipole

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Symmetries of Matter

NO SPIN (PAIRED SPIN) | **SPIN (UNPAIRED SPIN)**

TYPE	DIPOLE ALIGNMENT	EXAMPLES	TYPE	SPIN ALIGNMENT	EXAMPLES
ferrielectric	$\vec{P}_i\neq 0, \vec{P}\neq 0$	$DyMn_2O_5$	ferrimagnetic	$\vec{M}_i\neq 0, \vec{M}\neq 0$	Magnetite, iron garnet

SPACE – TIME SYMMETRY EXHIBITED IN MEDIA!

Spatial symmetry describes geometrical aspects.
Temporal symmetry describes spin / no – spin aspects.

THESE SYMMETRIES INFLUENCE MATERIAL TENSOR PROPERTIES!

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Symmetry and Effect on Constitutive Relations

$$\vec{D}(x,y,z,\omega)=\overleftrightarrow{\varepsilon}(\omega)\cdot\vec{E}(x,y,z,\omega)+\overleftrightarrow{\xi}(\omega)\cdot\vec{H}(x,y,z,\omega)$$
$$\vec{B}(x,y,z,\omega)=\overleftrightarrow{\zeta}(\omega)\cdot\vec{E}(x,y,z,\omega)+\overleftrightarrow{\mu}(\omega)\cdot\vec{H}(x,y,z,\omega)$$

$$x,y,z=Spatial\ \ Symmetries\ (\overbrace{Reflection}^{change\ 1\ coord.}\ /\ \overbrace{Rotation}^{change\ 2\ coord.}\ /\ \overbrace{Inversion}^{change\ 3\ coord.})$$

$$\vec{D},\vec{E}=\overbrace{Polar\ Vectors}^{\substack{maintained\ by\ charge/\\ atoms\ with\ no\ spin}}\ \ (handedness\ DOES\ NOT\ change\ on\ \begin{matrix}reflection\\ inversion\end{matrix})$$

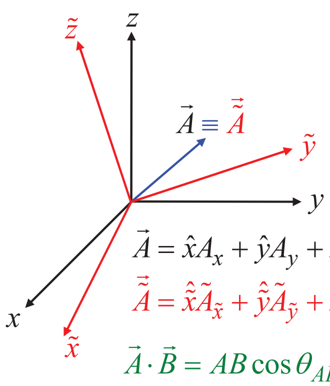
$$\vec{B},\vec{H}=\overbrace{Axial\ Vectors}^{\substack{maintained\ by\ current/\\ atoms\ with\ spin}}\ \ (handedness\ DOES\ change\ on\ \begin{matrix}reflection\\ inversion\end{matrix})$$

$$\overleftrightarrow{\varepsilon},\overleftrightarrow{\mu}=Polar\ Tensors\qquad \overleftrightarrow{\xi},\overleftrightarrow{\zeta}=Axial\ Tensors$$

$$t=Temporal\ Symmetry\ (reversal-manifested\ in\ \omega\ frequency\ domain)$$

> *Exactly how do symmetry operations influence form of* $\overleftrightarrow{\varepsilon},\overleftrightarrow{\mu},\overleftrightarrow{\xi},\overleftrightarrow{\zeta}$?
> *Need to understand transform matrices and these above concepts!*

Symmetry – Spatial Transformation Matrix



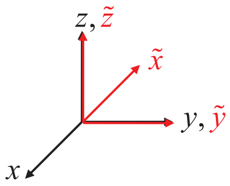
$$\vec{A}=\hat{x}A_x+\hat{y}A_y+\hat{z}A_z$$
$$\vec{\tilde{A}}=\hat{\tilde{x}}\tilde{A}_{\tilde{x}}+\hat{\tilde{y}}\tilde{A}_{\tilde{y}}+\hat{\tilde{z}}\tilde{A}_{\tilde{z}}$$
$$\vec{A}\cdot\vec{B}=AB\cos\theta_{AB}$$

$$\left.\begin{aligned}
\tilde{A}_{\tilde{x}}&=\hat{\tilde{x}}\cdot\overbrace{\vec{\tilde{A}}}^{\equiv\vec{A}}\equiv\hat{\tilde{x}}\cdot(\hat{x}A_x+\hat{y}A_y+\hat{z}A_z)\\
\tilde{A}_{\tilde{y}}&=\hat{\tilde{y}}\cdot\overbrace{\vec{\tilde{A}}}^{\equiv\vec{A}}\equiv\hat{\tilde{y}}\cdot(\hat{x}A_x+\hat{y}A_y+\hat{z}A_z)\\
\tilde{A}_{\tilde{z}}&=\hat{\tilde{z}}\cdot\overbrace{\vec{\tilde{A}}}^{\equiv\vec{A}}\equiv\hat{\tilde{z}}\cdot(\hat{x}A_x+\hat{y}A_y+\hat{z}A_z)
\end{aligned}\right\}\Rightarrow$$

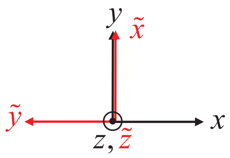
$$\overbrace{\begin{bmatrix}\tilde{A}_{\tilde{x}}\\ \tilde{A}_{\tilde{y}}\\ \tilde{A}_{\tilde{z}}\end{bmatrix}}^{\vec{\tilde{A}}}=\overbrace{\begin{bmatrix}\hat{\tilde{x}}\cdot\hat{x}&\hat{\tilde{x}}\cdot\hat{y}&\hat{\tilde{x}}\cdot\hat{z}\\ \hat{\tilde{y}}\cdot\hat{x}&\hat{\tilde{y}}\cdot\hat{y}&\hat{\tilde{y}}\cdot\hat{z}\\ \hat{\tilde{z}}\cdot\hat{x}&\hat{\tilde{z}}\cdot\hat{y}&\hat{\tilde{z}}\cdot\hat{z}\end{bmatrix}}^{\overleftrightarrow{T}}\overbrace{\begin{bmatrix}A_x\\ A_y\\ A_z\end{bmatrix}}^{\vec{A}}\quad or\quad \vec{\tilde{A}}=\overleftrightarrow{T}\cdot\vec{A}$$

> $\vec{\tilde{A}}=\overleftrightarrow{T}\cdot\vec{A}$ ... *generic point group transformation*
> *(easy to show that* $\vec{A}=\overleftrightarrow{T}^{-1}\cdot\vec{\tilde{A}}=\overleftrightarrow{T}^{T}\cdot\vec{\tilde{A}}$*)*

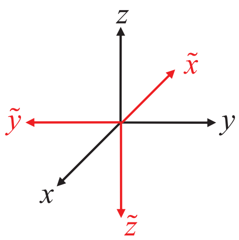
Symmetry – Spatial Transform Examples (see Appendix for more examples)



$$m_x = \overleftrightarrow{T}_{mx} = \begin{bmatrix} \hat{\tilde{x}}\cdot\hat{x} & \hat{\tilde{x}}\cdot\hat{y} & \hat{\tilde{x}}\cdot\hat{z} \\ \hat{\tilde{y}}\cdot\hat{x} & \hat{\tilde{y}}\cdot\hat{y} & \hat{\tilde{y}}\cdot\hat{z} \\ \hat{\tilde{z}}\cdot\hat{x} & \hat{\tilde{z}}\cdot\hat{y} & \hat{\tilde{z}}\cdot\hat{z} \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \; ...(rhs \to lhs)$$



$$4_z = \overleftrightarrow{T}_{4rz} = \begin{bmatrix} \hat{\tilde{x}}\cdot\hat{x} & \hat{\tilde{x}}\cdot\hat{y} & \hat{\tilde{x}}\cdot\hat{z} \\ \hat{\tilde{y}}\cdot\hat{x} & \hat{\tilde{y}}\cdot\hat{y} & \hat{\tilde{y}}\cdot\hat{z} \\ \hat{\tilde{z}}\cdot\hat{x} & \hat{\tilde{z}}\cdot\hat{y} & \hat{\tilde{z}}\cdot\hat{z} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \; ...(rhs \to rhs)$$



$$\bar{1} = \overleftrightarrow{T}_{\bar{1}} = \begin{bmatrix} \hat{\tilde{x}}\cdot\hat{x} & \hat{\tilde{x}}\cdot\hat{y} & \hat{\tilde{x}}\cdot\hat{z} \\ \hat{\tilde{y}}\cdot\hat{x} & \hat{\tilde{y}}\cdot\hat{y} & \hat{\tilde{y}}\cdot\hat{z} \\ \hat{\tilde{z}}\cdot\hat{x} & \hat{\tilde{z}}\cdot\hat{y} & \hat{\tilde{z}}\cdot\hat{z} \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = -\overleftrightarrow{I} \; ...(rhs \to lhs)$$

Note: Rotations are in a positive (right – hand rule) sense

Symmetry – Constitutive Relations

$$\vec{D}(x,y,z,\omega) = \overleftrightarrow{\varepsilon}(\omega)\cdot\vec{E}(x,y,z,\omega) + \overleftrightarrow{\xi}(\omega)\cdot\vec{H}(x,y,z,\omega)$$
$$\vec{B}(x,y,z,\omega) = \overleftrightarrow{\zeta}(\omega)\cdot\vec{E}(x,y,z,\omega) + \overleftrightarrow{\mu}(\omega)\cdot\vec{H}(x,y,z,\omega)$$

✓ x,y,z = *Spatial Symmetries* ($\overbrace{\textit{Reflection}}^{\text{change 1 coord.}}$ / $\overbrace{\textit{Rotation}}^{\text{change 2 coord.}}$ / $\overbrace{\textit{Inversion}}^{\text{change 3 coord.}}$)

$\vec{D},\vec{E}$ = $\overbrace{\textit{Polar Vectors}}^{\text{maintained by charge/ atoms with no spin}}$ (*handedness DOES NOT change on* reflection / inversion)

$\vec{B},\vec{H}$ = $\overbrace{\textit{Axial Vectors}}^{\text{maintained by current/ atoms with spin}}$ (*handedness DOES change on* reflection / inversion)

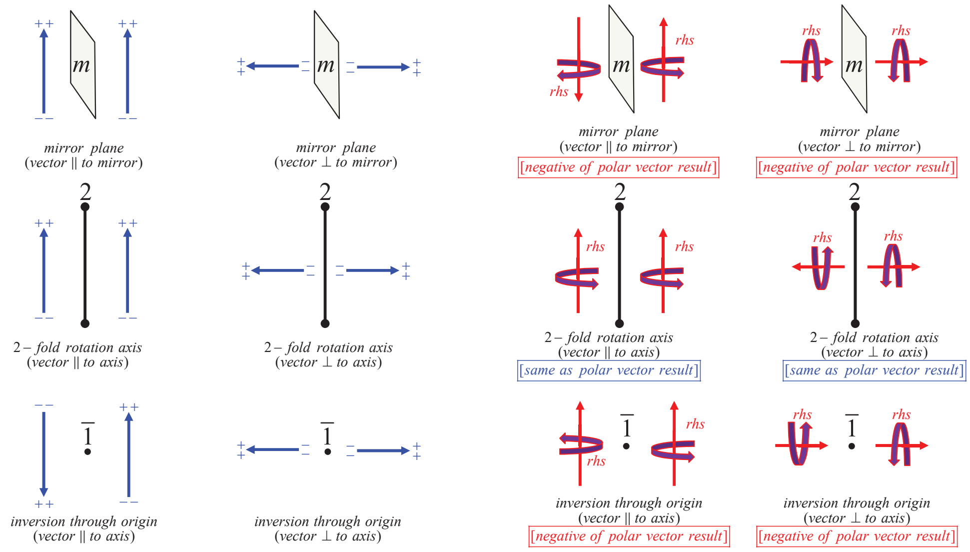
$\overleftrightarrow{\varepsilon},\overleftrightarrow{\mu}$ = *Polar Tensors* $\quad \overleftrightarrow{\xi},\overleftrightarrow{\zeta}$ = *Axial Tensors*

t = *Temporal Symmetry (reversal – manifested in ω frequency domain)*

Exactly how do symmetry operations influence form of $\overleftrightarrow{\varepsilon},\overleftrightarrow{\mu},\overleftrightarrow{\xi},\overleftrightarrow{\zeta}$?
Need to understand transform matrices and these above concepts!

Symmetry – Polar vs. Axial Vectors

POLAR VECTORS $(Non\text{-}handed,\ e.g.,\ \vec{E},\vec{D})$ *AXIAL VECTORS* $(Handed,\ e.g.,\ \vec{H},\vec{B})$



Symmetry – Polar vs. Axial Vectors

TRANSFORMATION CONCLUSION

$\tilde{\vec{E}} = \vec{T}\cdot\vec{E}$, $\tilde{\vec{D}} = \vec{T}\cdot\vec{D}$ ... *for polar vectors*

$\tilde{\vec{B}} = |\vec{T}|\,\vec{T}\cdot\vec{B}$, $\tilde{\vec{H}} = |\vec{T}|\,\vec{T}\cdot\vec{H}$ ... *for axial vectors*

$$\vec{T} = \begin{matrix} Spatial\ point\ transformation\ matrix \\ (mirror, rotation, inversion) \end{matrix} \quad , \quad |\vec{T}| = \begin{cases} -1 & ...mirror\ or\ inversion \\ +1 & ...rotation \end{cases}$$

$\vec{E},\vec{D},\vec{H},\vec{B}$ = *Fields in original coordinate system*

$\tilde{\vec{E}},\tilde{\vec{D}},\tilde{\vec{H}},\tilde{\vec{B}}$ = *Fields in transformed coordinate system*

Symmetry – Polar vs. Axial Tensors

$$\begin{aligned}\vec{D}(\vec{r},\omega)&=\vec{\vec{\varepsilon}}(\omega)\cdot\vec{E}(\vec{r},\omega)+\vec{\vec{\xi}}(\omega)\cdot\vec{H}(\vec{r},\omega)\\ \vec{B}(\vec{r},\omega)&=\vec{\vec{\zeta}}(\omega)\cdot\vec{E}(\vec{r},\omega)+\vec{\vec{\mu}}(\omega)\cdot\vec{H}(\vec{r},\omega)\end{aligned}\;\xrightarrow{\vec{\vec{T}}}\;\begin{aligned}\vec{\vec{T}}\cdot\vec{D}&=\vec{\vec{T}}\cdot\vec{\vec{\varepsilon}}\cdot\vec{E}+\vec{\vec{T}}\cdot\vec{\vec{\xi}}\cdot\vec{H}\\ \vec{\vec{T}}\cdot\vec{B}&=\vec{\vec{T}}\cdot\vec{\vec{\zeta}}\cdot\vec{E}+\vec{\vec{T}}\cdot\vec{\vec{\mu}}\cdot\vec{H}\end{aligned}\quad or$$

$$\begin{aligned}\overbrace{\vec{\vec{T}}\cdot\vec{D}}^{\tilde{\vec{D}}}&=\overbrace{\vec{\vec{T}}\cdot\vec{\vec{\varepsilon}}\cdot\vec{\vec{T}}^{-1}}^{\tilde{\vec{\vec{\varepsilon}}}}\cdot\overbrace{\vec{\vec{T}}\cdot\vec{E}}^{\tilde{\vec{E}}}+\overbrace{|\vec{\vec{T}}|\,\vec{\vec{T}}\cdot\vec{\vec{\xi}}\cdot\vec{\vec{T}}^{-1}}^{\tilde{\vec{\vec{\xi}}}}\cdot\overbrace{|\vec{\vec{T}}|\,\vec{\vec{T}}\cdot\vec{H}}^{\tilde{\vec{H}}}\\ \underbrace{|\vec{\vec{T}}|\,\vec{\vec{T}}\cdot\vec{B}}_{\tilde{\vec{B}}}&=\underbrace{|\vec{\vec{T}}|\,\vec{\vec{T}}\cdot\vec{\vec{\zeta}}\cdot\vec{\vec{T}}^{-1}}_{\tilde{\vec{\vec{\zeta}}}}\cdot\underbrace{\vec{\vec{T}}\cdot\vec{E}}_{\tilde{\vec{E}}}+\underbrace{\vec{\vec{T}}\cdot\vec{\vec{\mu}}\cdot\vec{\vec{T}}^{-1}}_{\tilde{\vec{\vec{\mu}}}}\cdot\underbrace{|\vec{\vec{T}}|\,\vec{\vec{T}}\cdot\vec{H}}_{\tilde{\vec{H}}}\end{aligned}\;\Rightarrow$$

$$\boxed{\begin{aligned}&\tilde{\vec{\vec{\varepsilon}}}=\vec{\vec{T}}\cdot\vec{\vec{\varepsilon}}\cdot\vec{\vec{T}}^{-1}\;,\;\tilde{\vec{\vec{\mu}}}=\vec{\vec{T}}\cdot\vec{\vec{\mu}}\cdot\vec{\vec{T}}^{-1}\quad ...Polar\ Tensors\\ &\tilde{\vec{\vec{\xi}}}=|\vec{\vec{T}}|\,\vec{\vec{T}}\cdot\vec{\vec{\xi}}\cdot\vec{\vec{T}}^{-1}\;,\;\tilde{\vec{\vec{\zeta}}}=|\vec{\vec{T}}|\,\vec{\vec{T}}\cdot\vec{\vec{\zeta}}\cdot\vec{\vec{T}}^{-1}\quad ...Axial\ Tensors\end{aligned}}\quad ...spatial\ transformation\ properties$$

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Symmetry – Constitutive Relations

$$\begin{aligned}\vec{D}(x,y,z,\omega)&=\vec{\vec{\varepsilon}}(\omega)\cdot\vec{E}(x,y,z,\omega)+\vec{\vec{\xi}}(\omega)\cdot\vec{H}(x,y,z,\omega)\\ \vec{B}(x,y,z,\omega)&=\vec{\vec{\zeta}}(\omega)\cdot\vec{E}(x,y,z,\omega)+\vec{\vec{\mu}}(\omega)\cdot\vec{H}(x,y,z,\omega)\end{aligned}$$

$x,y,z = Spatial\ Symmetries\ (\overbrace{Reflection}^{change\ 1\ coord.}\ /\ \overbrace{Rotation}^{change\ 2\ coord.}\ /\ \overbrace{Inversion}^{change\ 3\ coord.})$

$\vec{D},\vec{E} = \overbrace{Polar\ Vectors}^{maintained\ by\ charge/\ atoms\ with\ no\ spin}$ (handedness DOES NOT change on reflection/inversion)

$\vec{B},\vec{H} = \overbrace{Axial\ Vectors}^{maintained\ by\ current/\ atoms\ with\ spin}$ (handedness DOES change on reflection/inversion)

$\vec{\vec{\varepsilon}},\vec{\vec{\mu}} = Polar\ Tensors$ $\quad\vec{\vec{\xi}},\vec{\vec{\zeta}} = Axial\ Tensors$

$t = Temporal\ Symmetry$ (reversal – manifested in ω frequency domain)

> *Exactly how do symmetry operations influence form of* $\vec{\vec{\varepsilon}},\vec{\vec{\mu}},\vec{\vec{\xi}},\vec{\vec{\zeta}}$?
> *Need to understand transform matrices and these above concepts!*

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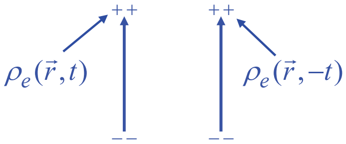
Symmetry – Time Reversal Symmetry

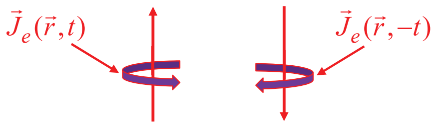
NO SPIN
(non-magnetic materials)



$t \qquad R\{t\} = -t$

$\rho_e(\vec{r},t) = \rho_e(\vec{r},-t)$
NO CHANGE

SPIN
(magnetic materials)



$t \qquad R\{t\} = -t$

$\vec{J}_e(\vec{r},t) = -\vec{J}_e(\vec{r},-t)$
REVERSAL

$R\{t\}$ (and often $'$) represents time reversal

KEY POINT *Time reversal useful for describing magnetic materials* *(e.g., ferromagnetic, antiferromagnetic, etc.)*

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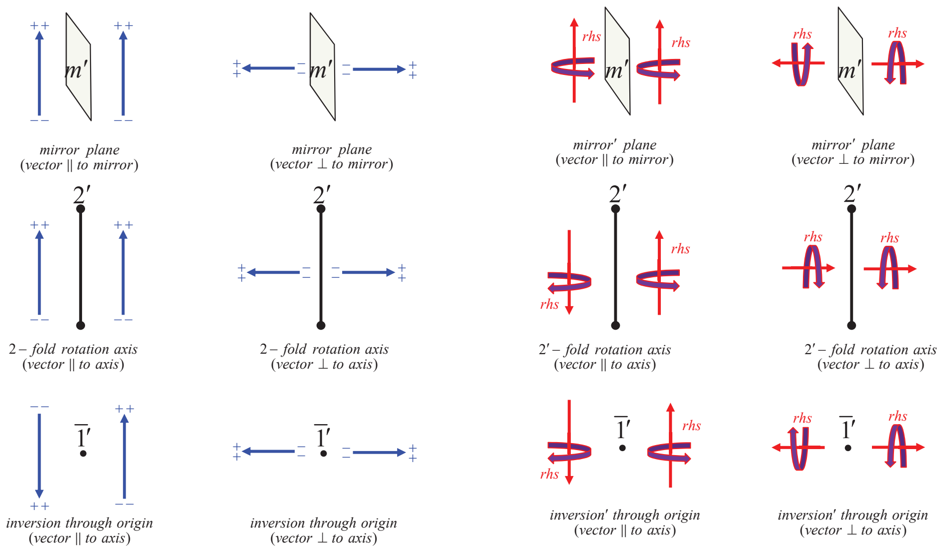
Symmetry – Space/Time Symmetry

POLAR VECTORS (with time inversion) *AXIAL VECTORS (with time inversion)*



NO CHANGE (Non − magnetic media) *REVERSAL (Magnetic media)*

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Symmetry – Transform Domain

$$\nabla\times\vec{E}(\vec{r},t)=-\vec{J}_h(\vec{r},t)-\frac{\partial\vec{B}(\vec{r},t)}{\partial t} \qquad \nabla\times\vec{E}(\vec{r},\omega)=-\vec{J}_h(\vec{r},\omega)-j\omega\vec{B}(\vec{r},\omega)$$
$$\nabla\times\vec{H}(\vec{r},t)=\vec{J}_e(\vec{r},t)+\frac{\partial\vec{D}(\vec{r},t)}{\partial t} \xrightarrow{F.T.} \nabla\times\vec{H}(\vec{r},\omega)=\vec{J}_e(\vec{r},\omega)+j\omega\vec{D}(\vec{r},\omega)$$
$$\nabla\cdot\vec{D}(\vec{r},t)=\rho_e(\vec{r},t) \qquad \nabla\cdot\vec{D}(\vec{r},\omega)=\rho_e(\vec{r},\omega)$$
$$\nabla\cdot\vec{B}(\vec{r},t)=\rho_h(\vec{r},t) \qquad \nabla\cdot\vec{B}(\vec{r},\omega)=\rho_h(\vec{r},\omega)$$

$$\nabla\times\vec{E}(\vec{r},-t)=-\vec{J}_h(\vec{r},-t)-\frac{\partial\vec{B}(\vec{r},-t)}{\partial(-t)} \qquad \nabla\times\vec{E}^*(\vec{r},\omega)=-\vec{J}_h^*(\vec{r},\omega)+j\omega\vec{B}^*(\vec{r},\omega)$$
$$\nabla\times\vec{H}(\vec{r},-t)=\vec{J}_e(\vec{r},-t)+\frac{\partial\vec{D}(\vec{r},-t)}{\partial(-t)} \xrightarrow{F.T.} \nabla\times\vec{H}^*(\vec{r},\omega)=\vec{J}_e^*(\vec{r},\omega)-j\omega\vec{D}^*(\vec{r},\omega)$$
$$\nabla\cdot\vec{D}(\vec{r},-t)=\rho_e(\vec{r},-t) \qquad \nabla\cdot\vec{D}^*(\vec{r},\omega)=\rho_e^*(\vec{r},\omega)$$
$$\nabla\cdot\vec{B}(\vec{r},-t)=\rho_h(\vec{r},-t) \qquad \nabla\cdot\vec{B}^*(\vec{r},\omega)=\rho_h^*(\vec{r},\omega)$$

Relations for time invariance of Maxwell's equations

$$\vec{E}(\vec{r},\omega),\ \vec{D}(\vec{r},\omega),\ \rho_e(\vec{r},\omega),\ \vec{J}_h(\vec{r},\omega) \xrightarrow{R=1'} \vec{E}^*(\vec{r},\omega),\ \vec{D}^*(\vec{r},\omega),\ \rho_e^*(\vec{r},\omega),\ \vec{J}_h^*(\vec{r},\omega)$$
$$\vec{H}(\vec{r},\omega),\ \vec{B}(\vec{r},\omega),\ \rho_h(\vec{r},\omega),\ \vec{J}_e(\vec{r},\omega) \xrightarrow{R=1'} -\vec{H}^*(\vec{r},\omega),\ -\vec{B}^*(\vec{r},\omega),\ -\rho_h^*(\vec{r},\omega),\ -\vec{J}_e^*(\vec{r},\omega)$$

$$\begin{matrix}\vec{D} \\ \vec{B}\end{matrix} \xrightarrow{R=1'} \begin{matrix}\vec{D}^*=(\overleftrightarrow{\varepsilon}\cdot\vec{E}+\overleftrightarrow{\xi}\cdot\vec{H})^* \overset{Onsager\,\dagger}{=} (\overleftrightarrow{\varepsilon}^T\cdot\vec{E}-\overleftrightarrow{\zeta}^T\cdot\vec{H})^* \\ -\vec{B}^*=-(\overleftrightarrow{\zeta}\cdot\vec{E}+\overleftrightarrow{\mu}\cdot\vec{H})^*=-(-\overleftrightarrow{\xi}^T\cdot\vec{E}+\overleftrightarrow{\mu}^T\cdot\vec{H})^*\end{matrix} \overset{conj.}{\Rightarrow} \begin{matrix}\vec{D}=\overleftrightarrow{\varepsilon}^T\cdot\vec{E}-\overleftrightarrow{\zeta}^T\cdot\vec{H} \\ \vec{B}=-\overleftrightarrow{\xi}^T\cdot\vec{E}+\overleftrightarrow{\mu}^T\cdot\vec{H}\end{matrix}$$

$$\Rightarrow \begin{matrix}\overbrace{\overleftrightarrow{T}\cdot\vec{D}}^{\tilde{\vec{D}}}=\overbrace{\overleftrightarrow{T}\cdot\overleftrightarrow{\varepsilon}^T\cdot\overleftrightarrow{T}^{-1}}^{\tilde{\overleftrightarrow{\varepsilon}}}\cdot\overbrace{\overleftrightarrow{T}\cdot\vec{E}}^{\tilde{\vec{E}}}\overbrace{-|\overleftrightarrow{T}|\overleftrightarrow{T}\cdot\overleftrightarrow{\zeta}^T\cdot\overleftrightarrow{T}^{-1}}^{\tilde{\overleftrightarrow{\xi}}}\cdot\overbrace{|\overleftrightarrow{T}|\overleftrightarrow{T}\cdot\vec{H}}^{\tilde{\vec{H}}} \\ \underbrace{|\overleftrightarrow{T}|\overleftrightarrow{T}\cdot\vec{B}}_{\tilde{\vec{B}}}=\underbrace{-|\overleftrightarrow{T}|\overleftrightarrow{T}\cdot\overleftrightarrow{\xi}^T\cdot\overleftrightarrow{T}^{-1}}_{\tilde{\overleftrightarrow{\zeta}}}\cdot\underbrace{\overleftrightarrow{T}\cdot\vec{E}}_{\tilde{\vec{E}}}+\underbrace{\overleftrightarrow{T}\cdot\overleftrightarrow{\mu}^T\cdot\overleftrightarrow{T}^{-1}}_{\tilde{\overleftrightarrow{\mu}}}\cdot\underbrace{|\overleftrightarrow{T}|\overleftrightarrow{T}\cdot\vec{H}}_{\tilde{\vec{H}}}\end{matrix} \rightarrow \boxed{\begin{matrix}\tilde{\overleftrightarrow{\varepsilon}}=\overleftrightarrow{T}\cdot\overleftrightarrow{\varepsilon}^T\cdot\overleftrightarrow{T}^{-1},\ \tilde{\overleftrightarrow{\mu}}=\overleftrightarrow{T}\cdot\overleftrightarrow{\mu}^T\cdot\overleftrightarrow{T}^{-1} \\ \tilde{\overleftrightarrow{\xi}}=-|\overleftrightarrow{T}|\overleftrightarrow{T}\cdot\overleftrightarrow{\zeta}^T\cdot\overleftrightarrow{T}^{-1},\ \tilde{\overleftrightarrow{\zeta}}=-|\overleftrightarrow{T}|\overleftrightarrow{T}\cdot\overleftrightarrow{\xi}^T\cdot\overleftrightarrow{T}^{-1}\end{matrix}}$$

...spatial – temporal transform properties

† *S. Tretyakov, A. Sihvola, and B. Jancewicz, "Onsager-Casimir Principle and the Constitutive Relations of Bi-Anisotropic Media", Journal of Electromagnetic Waves and Applications, vol. 16, no. 4, pp. 573-587, 2002.*

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Symmetry – Constitutive Relations

$$\vec{D}(x,y,z,\omega)=\overleftrightarrow{\varepsilon}(\omega)\cdot\vec{E}(x,y,z,\omega)+\overleftrightarrow{\xi}(\omega)\cdot\vec{H}(x,y,z,\omega)$$
$$\vec{B}(x,y,z,\omega)=\overleftrightarrow{\zeta}(\omega)\cdot\vec{E}(x,y,z,\omega)+\overleftrightarrow{\mu}(\omega)\cdot\vec{H}(x,y,z,\omega)$$

✓ $x,y,z=$ *Spatial Symmetries* ($\overbrace{\textit{Reflection}}^{\textit{change 1 coord.}}$ / $\overbrace{\textit{Rotation}}^{\textit{change 2 coord.}}$ / $\overbrace{\textit{Inversion}}^{\textit{change 3 coord.}}$)

✓ $\vec{D},\vec{E}=$ $\overbrace{\textit{Polar Vectors}}^{\textit{maintained by charge/ atoms with no spin}}$ (*handedness DOES NOT change on* $\begin{matrix}\textit{reflection}\\ \textit{inversion}\end{matrix}$)

✓ $\vec{B},\vec{H}=$ $\overbrace{\textit{Axial Vectors}}^{\textit{maintained by current/ atoms with spin}}$ (*handedness DOES change on* $\begin{matrix}\textit{reflection}\\ \textit{inversion}\end{matrix}$)

✓ $\overleftrightarrow{\varepsilon},\overleftrightarrow{\mu}=$ *Polar Tensors* ✓ $\overleftrightarrow{\xi},\overleftrightarrow{\zeta}=$ *Axial Tensors*

✓ $t=$ *Temporal Symmetry (reversal – manifested in ω frequency domain)*

Exactly how do symmetry operations influence form of $\overleftrightarrow{\varepsilon},\overleftrightarrow{\mu},\overleftrightarrow{\xi},\overleftrightarrow{\zeta}$?
Need to understand transform matrices and these above concepts!

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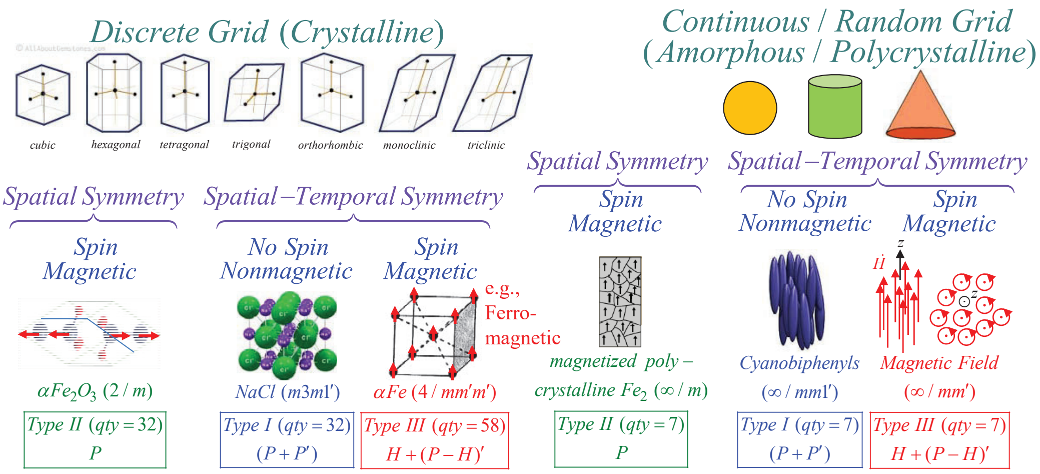
Symmetry – Types



$$\therefore 122\ \textit{Discrete} + 21\ \textit{Continuous Point Symmetry Groups} \rightarrow \vec{\vec{\varepsilon}}, \vec{\vec{\mu}}, \vec{\vec{\xi}}, \vec{\vec{\zeta}}$$

Symmetry – Point Groups

DISCRETE POINT GROUPS

Crystal Family (# of classes)	Nonmagnetic: Type I $P+P'$	Magnetic: Type II P	Magnetic: Type III $P(H)=H+(P-H)'$
Triclinic (5)	$11'$	1	
	$\bar{1}1'$	$\bar{1}$	$\bar{1}'$
Monoclinic (11)	$21'$	2	$2'$
	$m1'$	m	m'
	$2/m1'$	$2/m$	$2'/m'$ $2/m'$ $2'/m$
Orthorhombic (12)	$2221'$	222	$2'2'2$
	$mm21'$	$mm2$	$m'm'2$ $2'm'm$
	$mmm1'$	mmm	$mm'm'$ $m'm'm'$ mmm'
Tetragonal (31)	$41'$	4	$4'$
	$\bar{4}1'$	$\bar{4}$	$\bar{4}'$
	$4/m1'$	$4/m$	$4'/m$ $4/m'$ $4'/m'$
	$4221'$	422	$4'22'$ $42'2'$
	$4mm1'$	$4mm$	$4'mm'$ $4m'm'$
	$\bar{4}2m1'$	$\bar{4}2m$	$\bar{4}'2m'$ $\bar{4}'m2'$ $\bar{4}2'm'$
	$4/mmm1'$	$4/mmm$	$4'/mmm'$ $4/mm'm'$ $4/m'm'm'$ $4/m'mm$ $4'/m'm'm$
Trigonal (16)	$31'$	3	
	$\bar{3}1'$	$\bar{3}$	$\bar{3}'$
	$321'$	32	$32'$
	$3m1'$	$3m$	$3m'$
	$\bar{3}m1'$	$\bar{3}m$	$\bar{3}m'$ $\bar{3}'m'$ $\bar{3}'m$
Hexagonal (31)	$61'$	6	$6'$
	$\bar{6}1'$	$\bar{6}$	$\bar{6}'$
	$6/m1'$	$6/m$	$6'/m'$ $6/m'$ $6'/m$
	$6221'$	622	$6'22'$ $62'2'$
	$6mm1'$	$6mm$	$6'mm'$ $6m'm'$
	$\bar{6}m21'$	$\bar{6}m2$	$\bar{6}'2m'$ $\bar{6}'m2'$ $\bar{6}m'2'$
	$6/mmm1'$	$6/mmm$	$6'/m'mm'$ $6/mm'm'$ $6/m'm'm'$ $6/m'mm$ $6'/mmm'$
Cubic (16)	$231'$	23	
	$m\bar{3}1'$	$m\bar{3}$	$m'\bar{3}'$
	$4321'$	432	$4'32'$
	$\bar{4}3m1'$	$\bar{4}3m$	$\bar{4}'3m'$
	$m\bar{3}m1'$	$m\bar{3}m$	$m\bar{3}m'$ $m'\bar{3}'m'$ $m'\bar{3}'m$
	32 +	32 +	58 = 122 discrete groups

CONTINUOUS POINT GROUPS

Nonmagnetic: Type I $P+P'$	Magnetic: Type II P	Magnetic: Type III $P(H)=H+(P-H)'$
$\infty 1'$	∞	
$\infty/m1'$	∞/m	∞/m'
$\infty 21'$	$\infty 2$	$\infty 2'$
$\infty m1'$	∞m	$\infty m'$
$\infty/mm1'$	∞/mm	∞/mm' $\infty/m'm$ $\infty/m'm'$
$\infty\infty 1'$	$\infty\infty$	
$\infty\infty m1'$	$\infty\infty m$	$\infty\infty m'$
7 +	7 +	7 = 21 continuous groups

SPATIAL SYMMETRIES: Mirror, Rotation, Inversion through origin
TEMPORAL SYMMETRIES: Time Inversion (denoted by prime)

References:

A.Authier, *International Tables for Crystallography-Volume D Physical Properties of Crystals*, John Wiley, 2010.

V.Dmitriev, "Tables of the second rank constitutive tensors for linear homogeneous media described by the point magnetic groups of symmetry," *PIER* 28, 43 – 95, 2000.

A.Shubnikov and N. Belov, *Colored Symmetry*, Pergamon Press, 1964.

Marc De Graef, "Teaching crystallographic and magnetic point group symmetry using three-dimensional rendered visualizations," available at http://www.iucr.org/education/pamphlets/23

Note1: P' = Time − reversed point group symmetry elements *Note2: P = Point group symmetry elements* *Note3: H = Halving point subgroup of P*

Symmetry – Neumann's Principle

Symmetry group invariant under $\bar{\bar{T}} \Rightarrow$ material tensor invariant under $\bar{\bar{T}}$

$$\tilde{\bar{\bar{\varepsilon}}} = \bar{\bar{T}}\cdot\bar{\bar{\varepsilon}}\cdot\bar{\bar{T}}^{-1} = \bar{\bar{\varepsilon}} \;,\; \tilde{\bar{\bar{\mu}}} = \bar{\bar{T}}\cdot\bar{\bar{\mu}}\cdot\bar{\bar{T}}^{-1} = \bar{\bar{\mu}}$$
$$\tilde{\bar{\bar{\xi}}} = |\bar{\bar{T}}|\,\bar{\bar{T}}\cdot\bar{\bar{\xi}}\cdot\bar{\bar{T}}^{-1} = \bar{\bar{\xi}} \;,\; \tilde{\bar{\bar{\zeta}}} = |\bar{\bar{T}}|\,\bar{\bar{T}}\cdot\bar{\bar{\zeta}}\cdot\bar{\bar{T}}^{-1} = \bar{\bar{\zeta}} \quad or$$

$$\boxed{\begin{array}{c}\bar{\bar{T}}\cdot\bar{\bar{\varepsilon}} = \bar{\bar{\varepsilon}}\cdot\bar{\bar{T}} \;,\; \bar{\bar{T}}\cdot\bar{\bar{\mu}} = \bar{\bar{\mu}}\cdot\bar{\bar{T}} \\ \bar{\bar{T}}\cdot\bar{\bar{\xi}} = |\bar{\bar{T}}|\,\bar{\bar{\xi}}\cdot\bar{\bar{T}} \;,\; \bar{\bar{T}}\cdot\bar{\bar{\zeta}} = |\bar{\bar{T}}|\,\bar{\bar{\zeta}}\cdot\bar{\bar{T}}\end{array}} \;\ldots\; \begin{array}{c}Neumann's\\ Principle\\ (spatial)\end{array}$$

$$\tilde{\bar{\bar{\varepsilon}}} = \bar{\bar{T}}\cdot\bar{\bar{\varepsilon}}^{T}\cdot\bar{\bar{T}}^{-1} = \bar{\bar{\varepsilon}} \;,\; \tilde{\bar{\bar{\mu}}} = \bar{\bar{T}}\cdot\bar{\bar{\mu}}^{T}\cdot\bar{\bar{T}}^{-1} = \bar{\bar{\mu}}$$
$$\tilde{\bar{\bar{\xi}}} = -|\bar{\bar{T}}|\,\bar{\bar{T}}\cdot\bar{\bar{\zeta}}^{T}\cdot\bar{\bar{T}}^{-1} = \bar{\bar{\xi}} \;,\; \tilde{\bar{\bar{\zeta}}} = -|\bar{\bar{T}}|\,\bar{\bar{T}}\cdot\bar{\bar{\xi}}^{T}\cdot\bar{\bar{T}}^{-1} = \bar{\bar{\zeta}} \quad or$$

$$\boxed{\begin{array}{c}\bar{\bar{T}}\cdot\bar{\bar{\varepsilon}} = \bar{\bar{\varepsilon}}^{T}\cdot\bar{\bar{T}} \;,\; \bar{\bar{T}}\cdot\bar{\bar{\mu}} = \bar{\bar{\mu}}^{T}\cdot\bar{\bar{T}} \\ \bar{\bar{T}}\cdot\bar{\bar{\xi}} = -|\bar{\bar{T}}|\,\bar{\bar{\zeta}}^{T}\cdot\bar{\bar{T}} \;,\; \bar{\bar{T}}\cdot\bar{\bar{\zeta}} = -|\bar{\bar{T}}|\,\bar{\bar{\xi}}^{T}\cdot\bar{\bar{T}}\end{array}} \;\ldots\; \begin{array}{c}Neumann's\\ Principle\\ (spatial-temporal)\end{array}$$

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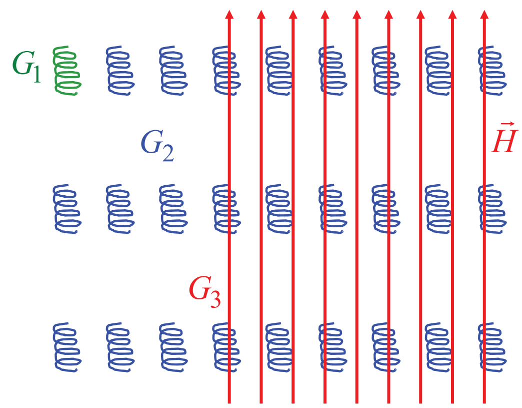
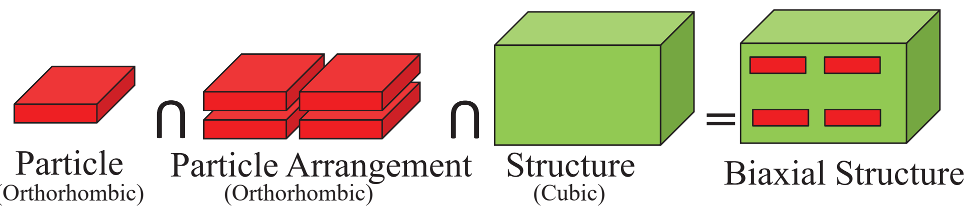
Symmetry – Curie's Principle



$$G_{total} = G_1 \bigcap G_2 \bigcap G_3 \bigcap \cdots G_n \cdots \bigcap G_N \quad \ldots Curie's\ Principle$$



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Symmetry – Neumann's Principle Example

$$\vec{\vec{T}} \cdot \vec{\vec{\varepsilon}} = \vec{\vec{\varepsilon}} \cdot \vec{\vec{T}} \quad \ldots (Neumann's\ Principle\ for\ group\ 4 = 4_z) \quad \Rightarrow$$

tetragonal

$$\vec{\vec{T}} = 4_z \Rightarrow \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ -\varepsilon_{xx} & -\varepsilon_{xy} & -\varepsilon_{xz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} = \begin{bmatrix} -\varepsilon_{xy} & \varepsilon_{xx} & \varepsilon_{xz} \\ -\varepsilon_{yy} & \varepsilon_{yx} & \varepsilon_{yz} \\ -\varepsilon_{zy} & \varepsilon_{zx} & \varepsilon_{zz} \end{bmatrix} \quad \therefore\ \vec{\vec{\varepsilon}} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & 0 \\ -\varepsilon_{xy} & \varepsilon_{xx} & 0 \\ 0 & 0 & \varepsilon_{zz} \end{bmatrix} \ldots from\ 9\ to\ 3!!!$$

$$\ldots similar\ analysis\ for\ \vec{\vec{\mu}}, \vec{\vec{\xi}}, \vec{\vec{\zeta}}$$

$$HOMEWORK:\ Find\ tensor\ form\ of\ \vec{\vec{\varepsilon}}, \vec{\vec{\mu}}, \vec{\vec{\xi}}, \zeta\ \ for\ symmetry\ group\ mmm1'\ (m_x, m_y, m_z, 1')$$

*Victor Dmitriev, "*Tables of the Second Rank Constitutive Tensors For Linear Homogeneous Media Described by the Point Magnetic Groups of Symmetry*," Progress in Electromagnetic Research, vol. 28, pp. 43-95, 2000.

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Symmetry – Tensor Influences Meas./Appl.

Application: Magnitude/Phase Control

Application: Polarization Control

$$\vec{\vec{\varepsilon}} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & 0 \\ -\varepsilon_{xy} & \varepsilon_{xx} & 0 \\ 0 & 0 & \varepsilon_{zz} \end{bmatrix}$$

Co-pol

Cross-pol

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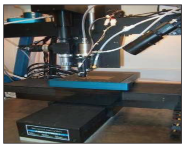
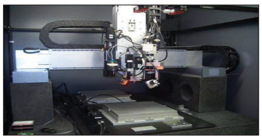
Motivation – Fabrication Capabilities (Infuse Symmetry Into Material Design)!

Technology Circa 2018!

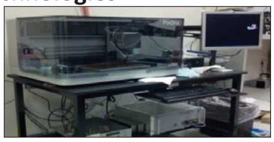
Additive Technologies* Dr. Keith Whites, Applied Research Associates.



M^{3}D



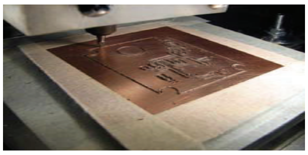
nScrypt



PixDro Inkjet

Connex 500 3D Printer

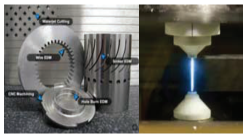
Subtractive Technologies



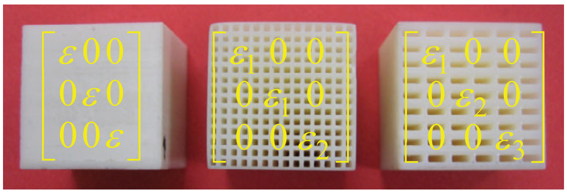
Milling Machine

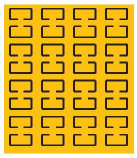
Chemical Etching

Photo Etching

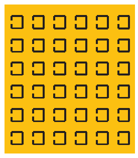


Wire EDM





?



?

which one is bianisotropic?

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Symmetry – Key Take-Aways!

KEY Take-Aways

Symmetry greatly influences material tensor structure and reduces the number of required measurements!!

Tensor structure tells us what co & cross polarization measurements are required!!

* * * *Symmetry can be infused into materials to control material tensor structure for desired applications!!!* * * *

...ultimately a key point!!!

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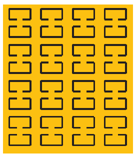
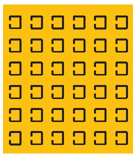
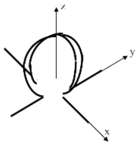
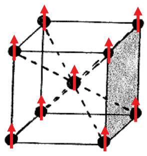
Factors that Influence Material Tensor Form – Homework

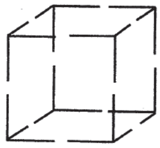
Determine the symmetry group for each object below.

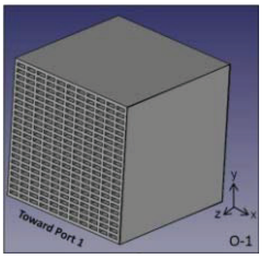


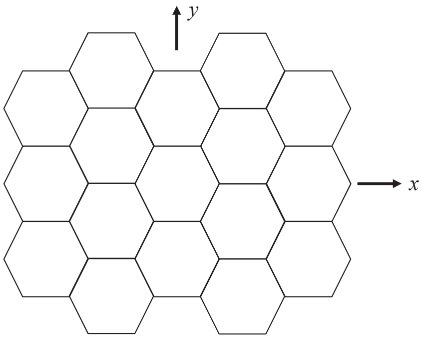


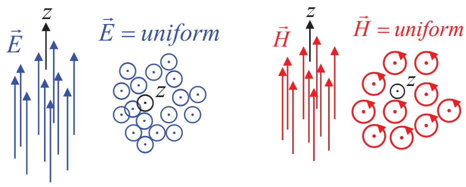












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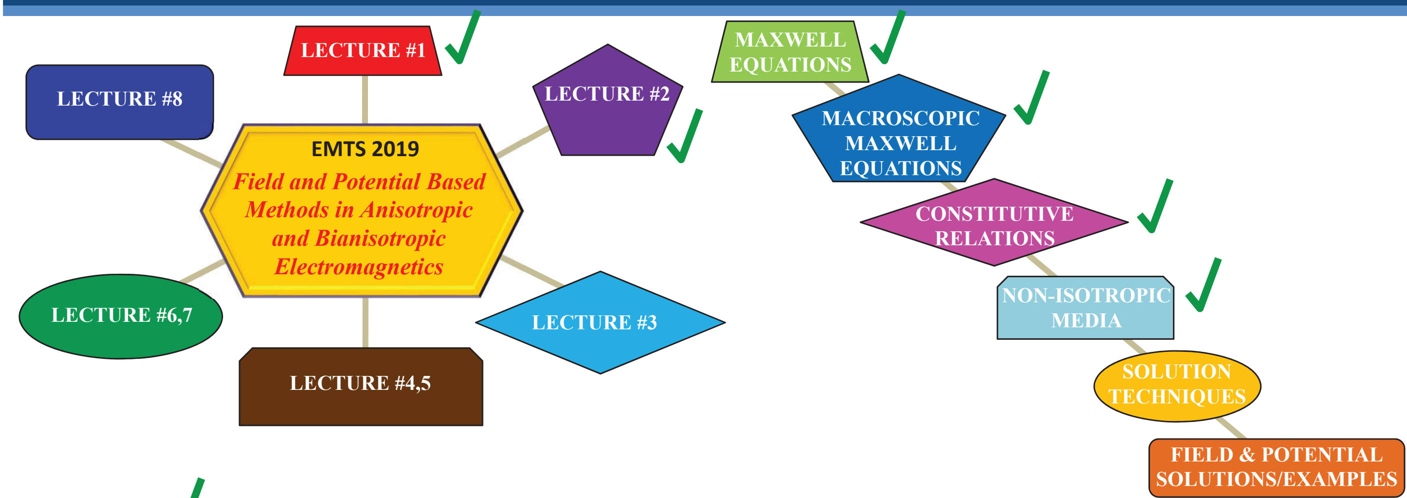
Overview – Lectures/Big Picture



LECTURE #1: Review various forms and regimes of validity of Maxwell's equations and constitutive relations.
LECTURE #2: Discuss factors that influence anisotropy and bianisotropy.
LECTURE #3: Discuss field and potential based methods for solving electromagnetic problems.
LECTURE #4,5: Provide several examples of field based solutions in complex media within and exterior to source regions.
LECTURE #6,7: Provide several examples of potential based solutions in complex media within and exterior to source regions.
LECTURE #8: Summary, conclusions and future research.

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Appendix – Symmetry Point Groups

DISCRETE POINT GROUPS

Crystal Family (# of classes)	Nonmagnetic: Type I $P+P'$	Magnetic: Type II P	Magnetic: Type III $P(H)=H+(P-H)'$
Triclinic (5)	$11'$ $\bar{1}1'$	1 $\bar{1}$	$\bar{1}'$
Monoclinic (11)	$21'$ $m1'$ $2/m1'$	2 m $2/m$	$2'$ m' $2'/m'$ $2/m'$ $2'/m$
Orthorhombic (12)	$2221'$ $mm21'$ $mmm1'$	222 $mm2$ mmm	$2'2'2$ $m'm'2$ $2'm'm$ $mm'm'$ $m'm'm'$ mmm'
Tetragonal (31)	$41'$ $\bar{4}1'$ $4/m1'$ $4221'$ $4mm1'$ $\bar{4}2m1'$ $4/mmm1'$	4 $\bar{4}$ $4/m$ 422 $4mm$ $\bar{4}2m$ $4/mmm$	$4'$ $\bar{4}'$ $4'/m$ $4/m'$ $4'/m'$ $4'22'$ $42'2'$ $4'mm'$ $4m'm'$ $\bar{4}'2m'$ $\bar{4}'m2'$ $\bar{4}2'm'$ $4'/mmm'$ $4/mm'm'$ $4/m'm'm'$ $4/m'mm$ $4'/m'm'm$
Trigonal (16)	$31'$ $\bar{3}1'$ $321'$ $3m1'$ $\bar{3}m1'$	3 $\bar{3}$ 32 $3m$ $\bar{3}m$	$\bar{3}'$ $32'$ $3m'$ $\bar{3}m'$ $\bar{3}'m'$ $\bar{3}'m$
Hexagonal (31)	$61'$ $\bar{6}1'$ $6/m1'$ $6221'$ $6mm1'$ $\bar{6}m21'$ $6/mmm1'$	6 $\bar{6}$ $6/m$ 622 $6mm$ $\bar{6}m2$ $6/mmm$	$6'$ $\bar{6}'$ $6'/m'$ $6/m'$ $6'/m$ $6'22'$ $62'2'$ $6'mm'$ $6m'm'$ $\bar{6}'2m'$ $\bar{6}'m2'$ $\bar{6}m'2'$ $6'/m'mm'$ $6/mm'm'$ $6/m'm'm'$ $6/m'mm$ $6'/mmm'$
Cubic (16)	$231'$ $m\bar{3}1'$ $4321'$ $\bar{4}3m1'$ $m\bar{3}m1'$	23 $m\bar{3}$ 432 $\bar{4}3m$ $m\bar{3}m$	$m'\bar{3}'$ $4'32'$ $\bar{4}'3m'$ $m\bar{3}m'$ $m'\bar{3}'m'$ $m'\bar{3}'m$
	32 +	32 +	58 = 122 discrete groups

CONTINUOUS POINT GROUPS

Nonmagnetic: Type I $P+P'$	Magnetic: Type II P	Magnetic: Type III $P(H)=H+(P-H)'$
$\infty1'$	∞	
$\infty/m1'$	∞/m	∞/m'
$\infty21'$	$\infty2$	$\infty2'$
$\infty m1'$	∞m	$\infty m'$
$\infty/mm1'$	∞/mm	∞/mm' $\infty/m'm$ $\infty/m'm'$
$\infty\infty1'$	$\infty\infty$	
$\infty\infty m1'$	$\infty\infty m$	$\infty\infty m'$
7 +	7 +	7 = 21 continuous groups

SPATIAL SYMMETRIES : Mirror, Rotation, Inversion through origin
TEMPORAL SYMMETRIES : Time Inversion (denoted by prime)

References:

A.Authier, *International Tables for Crystallography-Volume D Physical Properties of Crystals*, John Wiley, 2010.

V.Dmitriev, "Tables of the second rank constitutive tensors for linear homogeneous media described by the point magnetic groups of symmetry," *PIER* 28, 43–95, 2000.

A.Shubnikov and N. Belov, *Colored Symmetry*, Pergamon Press, 1964.

Marc De Graef, "Teaching crystallographic and magnetic point group symmetry using three-dimensional rendered visualizations," available at http://www.iucr.org/education/pamphlets/23

Note1 : P' = Time – reversed point group symmetry elements *Note2 : P = Point group symmetry elements* *Note3 : H = Halving point subgroup of P*

Appendix – Triclinic/Monoclinic Groups

Family	Point Group Symbol – IUC/Hermann–Mauguin: short	full	annotated	Schoenflies	Shubnikov	Generators	Symmetry Elements	[number]
Triclinic	$11'$	$11'$	$11'$	C_{1R}	$11'$	$1,1'$	$\{1\},\{1'\}$	[2] nc,p,nm
Triclinic	1	1	1	C_1	1	1	$\{1\}$	[1] nc,p,f
Triclinic	$\bar{1}1'$	$\bar{1}1'$	$\bar{1}1'$	$C_{iR}=S_{2R}$	$\tilde{2}1'$	$\bar{1},1'$	$\{1,\bar{1}\},\{1',\bar{1}'\}$	[4] c,np,nm
Triclinic	$\bar{1}$	$\bar{1}$	$\bar{1}$	$C_i=S_2$	$\tilde{2}$	$\bar{1}$	$\{1,\bar{1}\}$	[2] c,np,f
Triclinic	$\bar{1}'$	$\bar{1}'$	$\bar{1}'$	$C_i(C_1)$	$\tilde{2}'$	$\bar{1}'$	$\{1\},\{\bar{1}'\}$	[2] c,np,af
Monoclinic	$21'$	$1211'$	$2_y1'$	C_{2R}	$21'$	$2_y,1'$	$\{1,2_y\},\{1',2'_y\}$	[4] nc,p,nm
Monoclinic	2	121	2_y	C_2	2	2_y	$\{1,2_y\}$	[2] nc,p,f
Monoclinic	$2'$	$12'1$	$2'_y$	$C_2(C_1)$	$2'$	$2'_y$	$\{1\},\{2'_y\}$	[2] nc,p,f
Monoclinic	$m1'$	$1m11'$	m_y1'	$C_{sR}=C_{1hR}$	$m1'$	$m_y,1'$	$\{1,m_y\},\{1',m'_y\}$	[4] nc,p,nm
Monoclinic	m	$1m1$	m_y	$C_s=C_{1h}$	m	m_y	$\{1,m_y\}$	[2] nc,p,f
Monoclinic	m'	$1m'1$	m'_y	$C_s(C_1)$	m'	m'_y	$\{1\},\{m'_y\}$	[2] nc,p,f
Monoclinic	$2/m1'$	$1\frac{2}{m}11'$	$2_y/m_y1'$	C_{2hR}	$2:m1'$	$2_y,m_y,1'$	$\{1,\bar{1},2_y,m_y\},\{1',\bar{1}',2'_y,m'_y\}$	[8] c,np,nm
Monoclinic	$2/m$	$1\frac{2}{m}1$	$2_y/m_y$	C_{2h}	$2:m$	$2_y,m_y$	$\{1,\bar{1},2_y,m_y\}$	[4] c,np,f
Monoclinic	$2'/m'$	$1\frac{2'}{m'}1$	$2'_y/m'_y$	$C_{2h}(C_i)$	$2':m'$	$2'_y,m'_y$	$\{1,\bar{1}\},\{2'_y,m'_y\}$	[4] c,np,f
Monoclinic	$2/m'$	$1\frac{2}{m'}1$	$2_y/m'_y$	$C_{2h}(C_2)$	$2:m'$	$2_y,m'_y$	$\{1,2_y\},\{\bar{1}',m'_y\}$	[4] c,np,af
Monoclinic	$2'/m$	$1\frac{2'}{m}1$	$2'_y/m_y$	$C_{2h}(C_s)$	$2':m$	$2'_y,m_y$	$\{1,m_y\},\{\bar{1}',2'_y\}$	[4] c,np,af
Monoclinic	$21'$	$1121'$	$2_z1'$	C^z_{2R}	$21'$	$2_z,1'$	$\{1,2_z\},\{1',2'_z\}$	[4]
Monoclinic	2	112	2_z	C^z_2	2	2_z	$\{1,2_z\}$	[2]
Monoclinic	$2'$	$112'$	$2'_z$	$C^z_2(C_1)$	$2'$	$2'_z$	$\{1\},\{2'_z\}$	[2]
Monoclinic	$m1'$	$11m1'$	m_z1'	$C^z_{sR}=C^z_{1hR}$	$m1'$	$m_z,1'$	$\{1,m_z\},\{1',m'_z\}$	[4]
Monoclinic	m	$11m$	m_z	$C^z_s=C^z_{1h}$	m	m_z	$\{1,m_z\}$	[2]
Monoclinic	m'	$11m'$	m'_z	$C^z_s(C_1)$	m'	m'_z	$\{1\},\{m'_z\}$	[2]
Monoclinic	$2/m1'$	$11\frac{2}{m}1'$	$2_z/m_z1'$	C^z_{2hR}	$2:m1'$	$2_z,m_z,1'$	$\{1,\bar{1},2_z,m_z\},\{1',\bar{1}',2'_z,m'_z\}$	[8]
Monoclinic	$2/m$	$11\frac{2}{m}$	$2_z/m_z$	C^z_{2h}	$2:m$	$2_z,m_z$	$\{1,\bar{1},2_z,m_z\}$	[4]
Monoclinic	$2/m$	$\frac{2}{m}11$	$2_x/m_x$	C^x_{2h}	$2:m$	$2_x,m_x$	$\{1,\bar{1},2_x,m_x\}$	[4]
Monoclinic	$2'/m'$	$11\frac{2'}{m'}$	$2'_z/m'_z$	$C^z_{2h}(C_i)$	$2':m'$	$2'_z,m'_z$	$\{1,\bar{1}\},\{2'_z,m'_z\}$	[4]
Monoclinic	$2/m'$	$11\frac{2}{m'}$	$2_z/m'_z$	$C^z_{2h}(C^z_2)$	$2:m'$	$2_z,m'_z$	$\{1,2_z\},\{\bar{1}',m'_z\}$	[4]
Monoclinic	$2'/m$	$11\frac{2'}{m}$	$2'_z/m_z$	$C^z_{2h}(C^z_s)$	$2':m$	$2'_z,m_z$	$\{1,m_z\},\{\bar{1}',2'_z\}$	[4]

...Variations on IUC (last 12 rows)

EXTREMELY IMPORTANT!!! Symmetry elements of a given group can be found via the group generators

Generators + Neumann + Curie = Tensor Form

SPATIAL – TEMPORAL SYMMETRY ELEMENTS

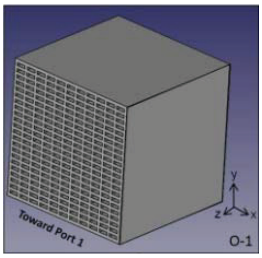
SPATIAL SYMMETRY ELEMENTS

Notes :
1. The symbol ' denotes time reversal
2. If a,b are elements of a group then :
   i. $a \times b = c$
   ii. $a' \times b = c'$ and $a \times b' = c'$
   iii. $a' \times b' = c$

Appendix – Orthorhombic Groups

Point Group Symbol; IUC / Hermann–Mauguin (short, full, annotated)

Family	short	full	annotated	Schoenflies	Shubnikov	Generators	Symmetry Elements	[number]
Orthorhombic	$2221'$	$2221'$	$2_x2_y2_z1'$	D_{2R}	$2:21'$	$2_x,2_y,1'$	$\{1,2_x,2_y,2_z\},\{1',2'_x,2'_y,2'_z\}$	[8] nc,np,nm
Orthorhombic	222	222	$2_x2_y2_z$	D_2	$2:2$	$2_x,2_y$	$\{1,2_x,2_y,2_z\}$	[4] nc,np,af
Orthorhombic	$2'2'2$	$2'2'2$	$2'_x2'_y2_z$	$D_2(C_2^z)$	$2:2'$	$2'_x,2'_y$	$\{1,2_z\},\{2'_x,2'_y\}$	[4] nc,np,f
Orthorhombic	$mm21'$	$mm21'$	$m_xm_y2_z1'$	C_{2vR}	$2\cdot m1'$	$m_x,2_z,1'$	$\{1,m_x,m_y,2_z\},\{1',m'_x,m'_y,2'_z\}$	[8] nc,p,nm
Orthorhombic	$mm2$	$mm2$	$m_xm_y2_z$	C_{2v}	$2\cdot m$	$m_x,2_z$	$\{1,m_x,m_y,2_z\}$	[4] nc,p,af
Orthorhombic	$m'm'2$	$m'm'2$	$m'_xm'_y2_z$	$C_{2v}(C_2^z)$	$2\cdot m'$	$m'_x,2_z$	$\{1,2_z\},\{m'_x,m'_y\}$	[4] nc,p,f
Orthorhombic	$2'm'm$	$2'm'm$	$2'_xm'_ym_z$	$C_{2v}^x(C_s^z)$	$2'\cdot m$	$2'_x,m_z$	$\{1,m_z\},\{2'_x,m'_y\}$ or $m'm2'$ $\{1,m_y\},\{m'_x,2'_z\}$	[4] nc,p,f
Orthorhombic	$mmm1'$	$\frac{2}{m}\frac{2}{m}\frac{2}{m}1'$	$m_xm_ym_z1'$	D_{2hR}	$m\cdot 2:m1'$	$m_x,m_y,m_z,1'$	$\{1,\bar{1},m_x,m_y,m_z,2_x,2_y,2_z\},\{1',\bar{1}',m'_x,m'_y,m'_z,2'_x,2'_y,2'_z\}$	[16] c,np,nm
Orthorhombic	mmm	$\frac{2}{m}\frac{2}{m}\frac{2}{m}$	$m_xm_ym_z$	D_{2h}	$m\cdot 2:m$	m_x,m_y,m_z	$\{1,\bar{1},m_x,m_y,m_z,2_x,2_y,2_z\}$	[8] c,np,af
Orthorhombic	$mm'm'$	$\frac{2}{m}\frac{2'}{m'}\frac{2'}{m'}$	$m_xm'_ym'_z$	$D_{2h}(C_{2h}^x)$	$m'\cdot 2:m$	m_x,m'_y,m'_z	$\{1,\bar{1},2_x,m_x\},\{2'_y,2'_z,m'_y,m'_z\}$ or $m'm'm$	[8] c,np,f
Orthorhombic	$m'm'm'$	$\frac{2}{m'}\frac{2}{m'}\frac{2}{m'}$	$m'_xm'_ym'_z$	$D_{2h}(D_2)$	$m'\cdot 2:m'$	m'_x,m'_y,m'_z	$\{1,2_x,2_y,2_z\},\{\bar{1}',m'_x,m'_y,m'_z\}$	[8] c,np,af
Orthorhombic	mmm'	$\frac{2'}{m}\frac{2'}{m}\frac{2}{m'}$	$m_xm_ym'_z$	$D_{2h}(C_{2v})$	$m\cdot 2:m'$	m_x,m_y,m'_z	$\{1,m_x,m_y,2_z\},\{\bar{1}',2'_x,2'_y,m'_z\}$ or $m'mm$	[8] c,np,af
VARIATIONS ON INTERNATIONAL UNION OF CRYTSTALLOGRAPHY								
Orthorhombic	$2mm$	$2mm$	$2_xm_ym_z$	C_{2v}^x	$m\cdot 2$	$2_x,m_z$	$\{1,2_x,m_y,m_z\}$	[4]



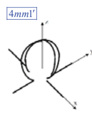
What symmetry group does this material belong?

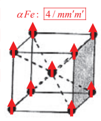
$\boxed{mmm1'}$

Appendix – Tetragonal Groups

Point Group Symbol; IUC / Hermann–Mauguin (short, full, annotated)

Family	short	full	annotated	Schoenflies	Shubnikov	Generators	Symmetry Elements	[number]
Tetragonal	$41'$	$41'$	$4_z1'$	C_{4R}	$41'$	$4_z,1'$	$\{1,2_z,\pm4_z\},\{1',2'_z,\pm4'_z\}$	[8]
Tetragonal	4	4	4_z	C_4	4	4_z	$\{1,2_z,\pm4_z\}$	[4]
Tetragonal	$4'$	$4'$	$4'_z$	$C_4(C_2^z)$	$4'$	$4'_z$	$\{1,2_z\},\{\pm4'_z\}$	[4]
Tetragonal	$\bar{4}1'$	$\bar{4}1'$	$\bar{4}_z1'$	S_{4R}	$\tilde{4}1'$	$\bar{4}_z,1'$	$\{1,2_z,\pm\bar{4}_z\},\{1',2'_z,\pm\bar{4}'_z\}$	[8]
Tetragonal	$\bar{4}$	$\bar{4}$	$\bar{4}_z$	S_4	$\tilde{4}$	$\bar{4}_z$	$\{1,2_z,\pm\bar{4}_z\}$	[4]
Tetragonal	$\bar{4}'$	$\bar{4}'$	$\bar{4}'_z$	$S_4(C_2^z)$	$\tilde{4}'$	$\bar{4}'_z$	$\{1,2_z\},\{\pm\bar{4}'_z\}$	[4]
Tetragonal	$4/m1'$	$\frac{4}{m}1'$	$4_z/m_z1'$	C_{4hR}	$4:m1'$	$4_z,m_z,1'$	$\{1,\bar{1},2_z,m_z,\pm4_z,\pm\bar{4}_z\},\{1',\bar{1}',2'_z,m'_z,\pm4'_z,\pm\bar{4}'_z\}$	[16]
Tetragonal	$4/m$	$\frac{4}{m}$	$4_z/m_z$	C_{4h}	$4:m$	$4_z,m_z$	$\{1,\bar{1},2_z,m_z,\pm4_z,\pm\bar{4}_z\}$	[8]
Tetragonal	$4'/m$	$\frac{4'}{m}$	$4'_z/m_z$	$C_{4h}(C_{2h}^z)$	$4':m$	$4'_z,m_z$	$\{1,\bar{1},2_z,m_z\},\{\pm4'_z,\pm\bar{4}'_z\}$	[8]
Tetragonal	$4/m'$	$\frac{4}{m'}$	$4_z/m'_z$	$C_{4h}(C_4)$	$4:m'$	$4_z,m'_z$	$\{1,2_z,\pm4_z\},\{\bar{1}',m'_z,\pm\bar{4}'_z\}$	[8]
Tetragonal	$4'/m'$	$\frac{4'}{m'}$	$4'_z/m'_z$	$C_{4h}(S_4)$	$4':m'$	$4'_z,m'_z$	$\{1,2_z,\pm\bar{4}_z\},\{\bar{1}',m'_z,\pm4'_z\}$	[8]
Tetragonal	$4221'$	$4221'$	$4_z2_x2_{xy}1'$	D_{4R}	$4:21'$	$4_z,2_x,1'$	$\{1,2_x,2_y,2_z,2_{xy},2_{\bar{x}y},\pm4_z\},\{1',2'_x,2'_y,2'_z,2'_{xy},2'_{\bar{x}y},\pm4'_z\}$	[16]
Tetragonal	422	422	$4_z2_x2_{xy}$	D_4	$4:2$	$4_z,2_x$	$\{1,2_x,2_y,2_z,2_{xy},2_{\bar{x}y},\pm4_z\}$	[8]
Tetragonal	$4'22'$	$4'22'$	$4'_z2_x2'_{xy}$	$D_4(D_2)$	$4':2$	$4'_z,2_x$	$\{1,2_x,2_y,2_z\},\{2'_{xy},2'_{\bar{x}y},\pm4'_z\}$	[8]
Tetragonal	$42'2'$	$42'2'$	$4_z2'_x2'_{xy}$	$D_4(C_4)$	$4:2'$	$4_z,2'_x$	$\{1,2_z,\pm4_z\},\{2'_x,2'_y,2'_{xy},2'_{\bar{x}y}\}$	[8]
Tetragonal	$4mm1'$	$4mm1'$	$4_zm_xm_{xy}1'$	C_{4vR}	$4\cdot m1'$	$4_z,m_x,1'$	$\{1,m_x,m_y,m_{xy},m_{\bar{x}y},2_z,\pm4_z\},\{1',m'_x,m'_y,m'_{xy},m'_{\bar{x}y},2'_z,\pm4'_z\}$	[16]
Tetragonal	$4mm$	$4mm$	$4_zm_xm_{xy}$	C_{4v}	$4\cdot m$	$4_z,m_x$	$\{1,m_x,m_y,m_{xy},m_{\bar{x}y},2_z,\pm4_z\}$	[8]
Tetragonal	$4'mm'$	$4'mm'$	$4'_zm_xm'_{xy}$	$C_{4v}(C_{2v})$	$4'\cdot m$	$4'_z,m_x$	$\{1,m_x,m_y,2_z\},\{m'_{xy},m'_{\bar{x}y},\pm4'_z\}$	[8]
Tetragonal	$4m'm'$	$4m'm'$	$4_zm'_xm'_{xy}$	$C_{4v}(C_4)$	$4\cdot m'$	$4_z,m'_x$	$\{1,2_z,\pm4_z\},\{m'_x,m'_y,m'_{xy},m'_{\bar{x}y}\}$	[8]
Tetragonal	$\bar{4}2m1'$	$\bar{4}2m1'$	$\bar{4}_z2_xm_{xy}1'$	D_{2dR}	$\bar{4}\cdot m1'$	$\bar{4}_z,2_x,1'$	$\{1,2_x,2_y,2_z,m_{xy},m_{\bar{x}y},\pm\bar{4}_z\},\{1',2'_x,2'_y,2'_z,m'_{xy},m'_{\bar{x}y},\pm\bar{4}'_z\}$	[16]
Tetragonal	$\bar{4}2m$	$\bar{4}2m$	$\bar{4}_z2_xm_{xy}$	D_{2d}	$\bar{4}\cdot m$	$\bar{4}_z,2_x$	$\{1,2_x,2_y,2_z,m_{xy},m_{\bar{x}y},\pm\bar{4}_z\}$	[8]
Tetragonal	$\bar{4}'2m'$	$\bar{4}'2m'$	$\bar{4}'_z2_xm'_{xy}$	$D_{2d}(D_2)$	$\bar{4}'\cdot m'$	$\bar{4}'_z,2_x$	$\{1,2_x,2_y,2_z\},\{m'_{xy},m'_{\bar{x}y},\pm\bar{4}'_z\}$	[8]
Tetragonal	$\bar{4}'m2'$	$\bar{4}'m2'$	$\bar{4}'_zm_x2'_{xy}$	$D_{2d}^x(C_{2v})$	$\bar{4}'\cdot m$	$\bar{4}'_z,m_x$	$\{1,m_x,m_y,2_z\},\{2'_{xy},2'_{\bar{x}y},\pm\bar{4}'_z\}$	[8]
Tetragonal	$\bar{4}2'm'$	$\bar{4}2'm'$	$\bar{4}_z2'_xm'_{xy}$	$D_{2d}(S_4)$	$\bar{4}\cdot m'$	$\bar{4}_z,2'_x$	$\{1,2_z,\pm\bar{4}_z\},\{2'_x,2'_y,m'_{xy},m'_{\bar{x}y}\}$	[8]
Tetragonal	$4/mmm1'$	$\frac{4}{m}\frac{2}{m}\frac{2}{m}1'$	$4_z/m_zm_xm_{xy}1'$	D_{4hR}	$m\cdot 4:m1'$	$4_z,m_z,m_x,1'$	$\{1,\bar{1},2_x,2_y,2_z,2_{xy},2_{\bar{x}y},m_x,m_y,m_z,m_{xy},m_{\bar{x}y},\pm4_z,\pm\bar{4}_z\},$ $\{1',\bar{1}',2'_x,2'_y,2'_z,2'_{xy},2'_{\bar{x}y},m'_x,m'_y,m'_z,m'_{xy},m'_{\bar{x}y},\pm4'_z,\pm\bar{4}'_z\}$	[32]
Tetragonal	$4/mmm$	$\frac{4}{m}\frac{2}{m}\frac{2}{m}$	$4_z/m_zm_xm_{xy}$	D_{4h}	$m\cdot 4:m$	$4_z,m_z,m_x$	$\{1,\bar{1},2_x,2_y,2_z,2_{xy},2_{\bar{x}y},m_x,m_y,m_z,m_{xy},m_{\bar{x}y},\pm4_z,\pm\bar{4}_z\}$	[16]
Tetragonal	$4'/mmm'$	$\frac{4'}{m}\frac{2}{m}\frac{2'}{m'}$	$4'_z/m_zm_xm'_{xy}$	$D_{4h}(D_{2h})$	$m\cdot 4':m$	$4'_z,m_z,m_x$	$\{1,\bar{1},2_x,2_y,2_z,m_x,m_y,m_z\},\{2'_{xy},2'_{\bar{x}y},m'_{xy},m'_{\bar{x}y},\pm4'_z,\pm\bar{4}'_z\}$	[16]
Tetragonal	$4/mm'm'$	$\frac{4}{m}\frac{2'}{m'}\frac{2'}{m'}$	$4_z/m_zm'_xm'_{xy}$	$D_{4h}(C_{4h})$	$m'\cdot 4:m$	$4_z,m_z,m'_x$	$\{1,\bar{1},2_z,m_z,\pm4_z,\pm\bar{4}_z\},\{2'_x,2'_y,2'_{xy},2'_{\bar{x}y},m'_x,m'_y,m'_{xy},m'_{\bar{x}y}\}$	[16]
Tetragonal	$4/m'm'm'$	$\frac{4}{m'}\frac{2}{m'}\frac{2}{m'}$	$4_z/m'_zm'_xm'_{xy}$	$D_{4h}(D_4)$	$m'\cdot 4:m'$	$4_z,m'_z,m'_x$	$\{1,2_x,2_y,2_z,2_{xy},2_{\bar{x}y},\pm4_z\},\{\bar{1}',m'_x,m'_y,m'_z,m'_{xy},m'_{\bar{x}y},\pm\bar{4}'_z\}$	[16]
Tetragonal	$4/m'mm$	$\frac{4}{m'}\frac{2'}{m}\frac{2'}{m}$	$4_z/m'_zm_xm_{xy}$	$D_{4h}(C_{4v})$	$m\cdot 4:m'$	$4_z,m'_z,m_x$	$\{1,m_x,m_y,m_{xy},m_{\bar{x}y},2_z,\pm4_z\},\{\bar{1}',2'_x,2'_y,2'_{xy},2'_{\bar{x}y},m'_z,\pm\bar{4}'_z\}$	[16]
Tetragonal	$4'/m'm'm$	$\frac{4'}{m'}\frac{2}{m'}\frac{2'}{m}$	$4'_z/m'_zm'_xm_{xy}$	$D_{4h}(D_{2d})$	$m\cdot 4':m'$	$4'_z,m'_z,m'_x$	$\{1,2_x,2_y,2_z,m_{xy},m_{\bar{x}y},\pm\bar{4}_z\},\{\bar{1}',2'_{xy},2'_{\bar{x}y},m'_x,m'_y,m'_z,\pm4'_z\}$	[16]
VARIATIONS ON INTERNATIONAL UNION OF CRYTSTALLOGRAPHY								
Tetragonal	$\bar{4}m2$	$\bar{4}m2$	$\bar{4}_zm_x2_{xy}$	D_{2d}^x	$\bar{4}\cdot m$	$\bar{4}_z,m_x$	$\{1,2_{xy},2_{\bar{x}y},2_z,m_x,m_y,\pm\bar{4}_z\}$	





Appendix – Trigonal Groups

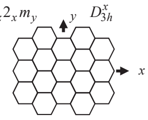
	Point Group Symbol							
	IUC/Hermann–Mauguin							
Family	*short*	*full*	*annotated*	*Schoenflies*	*Shubnikov*	*Generators*	*Symmetry Elements*	[*number*]
Trigonal	$31'$	$31'$	$3_z 1'$	C_{3R}	$31'$	$3_z, 1'$	$\{1, \pm 3_z\}, \{1', \pm 3'_z\}$	[6]
Trigonal	3	3	3_z	C_3	3	3_z	$\{1, \pm 3_z\}$	[3]
Trigonal	$\bar{3}1'$	$\bar{3}1'$	$\bar{3}_z 1'$	S_{6R}	$\tilde{6}1'$	$\bar{3}_z, 1'$	$\{1, \bar{1}, \pm 3_z, \pm \bar{3}_z\}, \{1', \bar{1}', \pm 3'_z, \pm \bar{3}'_z\}$	[12]
Trigonal	$\bar{3}$	$\bar{3}$	$\bar{3}_z$	S_6	$\tilde{6}$	$\bar{3}_z$	$\{1, \bar{1}, \pm 3_z, \pm \bar{3}_z\}$	[6]
Trigonal	$\bar{3}'$	$\bar{3}'$	$\bar{3}'_z$	$S_6(C_3)$	$\tilde{6}'$	$\bar{3}'_z$	$\{1, \pm 3_z\}, \{\bar{1}', \pm \bar{3}'_z\}$	[6]
Trigonal	$321'$	$3211'$	$3_z 2_x 1'$	D_{3R}	$3:21'$	$3_z, 2_x, 1'$	$\{1, 2_x, 2_{x\sqrt{3}y}, 2_{\bar{x}\sqrt{3}y}, \pm 3_z\}, \{1', 2'_x, 2'_{x\sqrt{3}y}, 2'_{\bar{x}\sqrt{3}y}, \pm 3'_z\}$	[12]
Trigonal	32	321	$3_z 2_x$	D_3	$3:2$	$3_z, 2_x$	$\{1, 2_x, 2_{x\sqrt{3}y}, 2_{\bar{x}\sqrt{3}y}, \pm 3_z\}$	[6]
Trigonal	$32'$	$32'1$	$3_z 2'_x$	$D_3(C_3)$	$3:2'$	$3_z, 2'_x$	$\{1, \pm 3_z\}, \{2'_x, 2'_{x\sqrt{3}y}, 2'_{\bar{x}\sqrt{3}y}\}$	[6]
Trigonal	$3m1'$	$3m11'$	$3_z m_x 1'$	C_{3vR}	$3 \cdot m1'$	$3_z, m_x, 1'$	$\{1, m_x, m_{x\sqrt{3}y}, m_{\bar{x}\sqrt{3}y}, \pm 3_z\}, \{1, m'_x, m'_{x\sqrt{3}y}, m'_{\bar{x}\sqrt{3}y}, \pm 3'_z\}$	[12]
Trigonal	$3m$	$3m1$	$3_z m_x$	C_{3v}	$3 \cdot m$	$3_z, m_x$	$\{1, m_x, m_{x\sqrt{3}y}, m_{\bar{x}\sqrt{3}y}, \pm 3_z\}$	[6]
Trigonal	$3m'$	$3m'1$	$3_z m'_x$	$C_{3v}(C_3)$	$3 \cdot m'$	$3_z, m'_x$	$\{1, \pm 3_z\}, \{m'_x, m'_{x\sqrt{3}y}, m'_{\bar{x}\sqrt{3}y}\}$	[6]
Trigonal	$\bar{3}m1'$	$\bar{3}\frac{2}{m}11'$	$\bar{3}_z m_x 1'$	D_{3dR}	$\tilde{6} \cdot m1'$	$\bar{3}_z, m_x, 1'$	$\{1, \bar{1}, 2_x, 2_{x\sqrt{3}y}, 2_{\bar{x}\sqrt{3}y}, m_x, m_{x\sqrt{3}y}, m_{\bar{x}\sqrt{3}y}, \pm 3_z, \pm \bar{3}_z\},$ $\{1', \bar{1}', 2'_x, 2'_{x\sqrt{3}y}, 2'_{\bar{x}\sqrt{3}y}, m'_x, m'_{x\sqrt{3}y}, m'_{\bar{x}\sqrt{3}y}, \pm 3'_z, \pm \bar{3}'_z\}$	[24]
Trigonal	$\bar{3}m$	$\bar{3}\frac{2}{m}1$	$\bar{3}_z m_x$	D_{3d}	$\tilde{6} \cdot m$	$\bar{3}_z, m_x$	$\{1, \bar{1}, 2_x, 2_{x\sqrt{3}y}, 2_{\bar{x}\sqrt{3}y}, m_x, m_{x\sqrt{3}y}, m_{\bar{x}\sqrt{3}y}, \pm 3_z, \pm \bar{3}_z\}$	[12]
Trigonal	$\bar{3}m'$	$\bar{3}\frac{2'}{m'}1$	$\bar{3}_z m'_x$	$D_{3d}(S_6)$	$\tilde{6} \cdot m'$	$\bar{3}_z, m'_x$	$\{1, \bar{1}, \pm 3_z, \pm \bar{3}_z\}, \{2'_x, 2'_{x\sqrt{3}y}, 2'_{\bar{x}\sqrt{3}y}, m'_x, m'_{x\sqrt{3}y}, m'_{\bar{x}\sqrt{3}y}\}$	[12]
Trigonal	$\bar{3}'m'$	$\bar{3}'\frac{2}{m'}1$	$\bar{3}'_z m'_x$	$D_{3d}(D_3)$	$\tilde{6}' \cdot m'$	$\bar{3}'_z, m'_x$	$\{1, 2_x, 2_{x\sqrt{3}y}, 2_{\bar{x}\sqrt{3}y}, \pm 3_z\}, \{\bar{1}', m'_x, m'_{x\sqrt{3}y}, m'_{\bar{x}\sqrt{3}y}, \pm \bar{3}'_z\}$	[12]
Trigonal	$\bar{3}'m$	$\bar{3}'\frac{2'}{m}1$	$\bar{3}'_z m_x$	$D_{3d}(C_{3v})$	$\tilde{6}' \cdot m$	$\bar{3}'_z, m_x$	$\{1, m_x, m_{x\sqrt{3}y}, m_{\bar{x}\sqrt{3}y}, \pm 3_z\}, \{\bar{1}', 2'_x, 2'_{x\sqrt{3}y}, 2'_{\bar{x}\sqrt{3}y}, \pm \bar{3}'_z\}$	[12]

Appendix – Hexagonal Groups

	Point Group Symbol							
	IUC/Hermann–Mauguin							
Family	*short*	*full*	*annotated*	*Schoenflies*	*Shubnikov*	*Generators*	*Symmetry Elements*	[*number*]
Hexagonal	$61'$	$61'$	$6_z 1'$	C_{6R}	$61'$	$6_z, 1'$	$\{1, 2_z, \pm 3_z, \pm 6_z\}, \{1', 2'_z, \pm 3'_z, \pm 6'_z\}$	[12]
Hexagonal	6	6	6_z	C_6	6	6_z	$\{1, 2_z, \pm 3_z, \pm 6_z\}$	[6]
Hexagonal	$6'$	$6'$	$6'_z$	$C_6(C_3)$	$6'$	$6'_z$	$\{1, \pm 3_z\}, \{2'_z, \pm 6'_z\}$	[6]
Hexagonal	$\bar{6}1'$	$\bar{6}1'$	$\bar{6}_z 1'$	C_{3hR}	$3:m1'$	$\bar{6}_z, 1'$	$\{1, m_z, \pm 3_z, \pm \bar{6}_z\}, \{1', m'_z, \pm 3'_z, \pm \bar{6}'_z\}$	[12]
Hexagonal	$\bar{6}$	$\bar{6}$	$\bar{6}_z$	C_{3h}	$3:m$	$\bar{6}_z$	$\{1, m_z, \pm 3_z, \pm \bar{6}_z\}$	[6]
Hexagonal	$\bar{6}'$	$\bar{6}'$	$\bar{6}'_z$	$C_{3h}(C_3)$	$3:m'$	$\bar{6}'_z$	$\{1, \pm 3_z\}, \{m'_z, \pm \bar{6}'_z\}$	[6]
Hexagonal	$6/m1'$	$\frac{6}{m}1'$	$6_z/m_z 1'$	C_{6hR}	$6:m1'$	$6_z, m_z, 1'$	$\{1, \bar{1}, 2_z, m_z, \pm 3_z, \pm \bar{3}_z, \pm 6_z, \pm \bar{6}_z\},$ $\{1', \bar{1}', 2'_z, m'_z, \pm 3'_z, \pm \bar{3}'_z, \pm 6'_z, \pm \bar{6}'_z\}$	[24]
Hexagonal	$6/m$	$\frac{6}{m}$	$6_z/m_z$	C_{6h}	$6:m$	$6_z, m_z$	$\{1, \bar{1}, 2_z, m_z, \pm 3_z, \pm \bar{3}_z, \pm 6_z, \pm \bar{6}_z\}$	[12]
Hexagonal	$6'/m'$	$\frac{6'}{m'}$	$6'_z/m'_z$	$C_{6h}(S_6)$	$6':m'$	$6'_z, m'_z$	$\{1, \bar{1}, \pm 3_z, \pm \bar{3}_z\}, \{2'_z, m'_z, \pm 6'_z, \pm \bar{6}'_z\}$	[12]
Hexagonal	$6/m'$	$\frac{6}{m'}$	$6_z/m'_z$	$C_{6h}(C_6)$	$6:m'$	$6_z, m'_z$	$\{1, 2_z, \pm 3_z, \pm 6_z\}, \{\bar{1}', m'_z, \pm \bar{3}'_z, \pm \bar{6}'_z\}$	[12]
Hexagonal	$6'/m$	$\frac{6'}{m}$	$6'_z/m_z$	$C_{6h}(C_{3h})$	$6':m$	$6'_z, m_z$	$\{1, m_z, \pm 3_z, \pm \bar{6}_z\}, \{\bar{1}', 2'_z, \pm \bar{3}'_z, \pm 6'_z\}$	[12]
Hexagonal	$6221'$	$6221'$	$6_z 2_x 2_y 1'$	D_{6R}	$6:21'$	$6_z, 2_x, 1'$	$\{1, 2_x, 2_{x\sqrt{3}y}, 2_{\bar{x}\sqrt{3}y}, 2_y, 2_{\sqrt{3}xy}, 2_{\sqrt{3}x\bar{y}}, 2_z, \pm 3_z, \pm 6_z\},$ $\{1', 2'_x, 2'_{x\sqrt{3}y}, 2'_{\bar{x}\sqrt{3}y}, 2'_y, 2'_{\sqrt{3}xy}, 2'_{\sqrt{3}x\bar{y}}, 2'_z, \pm 3'_z, \pm 6'_z\}$	[24]
Hexagonal	622	622	$6_z 2_x 2_y$	D_6	$6:2$	$6_z, 2_x$	$\{1, 2_x, 2_{x\sqrt{3}y}, 2_{\bar{x}\sqrt{3}y}, 2_y, 2_{\sqrt{3}xy}, 2_{\sqrt{3}x\bar{y}}, 2_z, \pm 3_z, \pm 6_z\}$	[12]
Hexagonal	$6'22'$	$6'22'$	$6'_z 2_x 2'_y$	$D_6(D_3)$	$6':2$	$6'_z, 2_x$	$\{1, 2_x, 2_{x\sqrt{3}y}, 2_{\bar{x}\sqrt{3}y}, \pm 3_z\}, \{2'_y, 2'_{\sqrt{3}xy}, 2'_{\sqrt{3}x\bar{y}}, 2'_z, \pm 6'_z\}$	[12]
Hexagonal	$62'2'$	$62'2'$	$6_z 2'_x 2'_y$	$D_6(C_6)$	$6:2'$	$6_z, 2'_x$	$\{1, 2_z, \pm 3_z, \pm 6_z\}, \{2'_x, 2'_{x\sqrt{3}y}, 2'_{\bar{x}\sqrt{3}y}, 2'_y, 2'_{\sqrt{3}xy}, 2'_{\sqrt{3}x\bar{y}}\}$	[12]
Hexagonal	$6mm1'$	$6mm1'$	$6_z m_x m_y 1'$	C_{6vR}	$6 \cdot m1'$	$6_z, m_x, 1'$	$\{1, m_x, m_{x\sqrt{3}y}, m_{\bar{x}\sqrt{3}y}, m_y, m_{\sqrt{3}xy}, m_{\sqrt{3}x\bar{y}}, 2_z, \pm 3_z, \pm 6_z\},$ $\{1', m'_x, m'_{x\sqrt{3}y}, m'_{\bar{x}\sqrt{3}y}, m'_y, m'_{\sqrt{3}xy}, m'_{\sqrt{3}x\bar{y}}, 2'_z, \pm 3'_z, \pm 6'_z\}$	[24]
Hexagonal	$6mm$	$6mm$	$6_z m_x m_y$	C_{6v}	$6 \cdot m$	$6_z, m_x$	$\{1, m_x, m_{x\sqrt{3}y}, m_{\bar{x}\sqrt{3}y}, m_y, m_{\sqrt{3}xy}, m_{\sqrt{3}x\bar{y}}, 2_z, \pm 3_z, \pm 6_z\}$	[12]
Hexagonal	$6'mm'$	$6'mm'$	$6'_z m_x m'_y$	$C_{6v}(C_{3v})$	$6' \cdot m$	$6'_z, m_x$	$\{1, m_x, m_{x\sqrt{3}y}, m_{\bar{x}\sqrt{3}y}, \pm 3_z\}, \{m'_y, m'_{\sqrt{3}xy}, m'_{\sqrt{3}x\bar{y}}, 2'_z, \pm 6'_z\}$	[12]
Hexagonal	$6m'm'$	$6m'm'$	$6_z m'_x m'_y$	$C_{6v}(C_6)$	$6 \cdot m'$	$6_z, m'_x$	$\{1, 2_z, \pm 3_z, \pm 6_z\}, \{m'_x, m'_{x\sqrt{3}y}, m'_{\bar{x}\sqrt{3}y}, m'_y, m'_{\sqrt{3}xy}, m'_{\sqrt{3}x\bar{y}}\}$	[12]

Appendix – Hexagonal Groups (cont'd)

Family	Point Group Symbol – IUC/Hermann–Mauguin: short	full	annotated	Schoenflies	Shubnikov	Generators	Symmetry Elements	[number]
Hexagonal	$\bar{6}m21'$	$\bar{6}m21'$	$\bar{6}_z m_x 2_y 1'$	D_{3hR}	$m\cdot 3:m1'$	$\bar{6}_z, m_x, 1'$	$\{1,2_y,2_{\sqrt{3}xy},2_{\sqrt{3}x\bar{y}},m_x,m_{x\sqrt{3}y},m_{\bar{x}\sqrt{3}y},m_z,\pm 3_z,\pm\bar{6}_z\},$ $\{1',2'_y,2'_{\sqrt{3}xy},2'_{\sqrt{3}x\bar{y}},m'_x,m'_{x\sqrt{3}y},m'_{\bar{x}\sqrt{3}y},m'_z,\pm 3'_z,\pm\bar{6}'_z\}$	[24]
Hexagonal	$\bar{6}m2$	$\bar{6}m2$	$\bar{6}_z m_x 2_y$	D_{3h}	$m\cdot 3:m$	$\bar{6}_z, m_x$	$\{1,2_y,2_{\sqrt{3}xy},2_{\sqrt{3}x\bar{y}},m_x,m_{x\sqrt{3}y},m_{\bar{x}\sqrt{3}y},m_z,\pm 3_z,\pm\bar{6}_z\}$	[12]
Hexagonal	$\bar{6}'2m'$	$\bar{6}'2m'$	$\bar{6}'_z 2_x m'_y$	$D^x_{3h}(D_3)$	$m'\cdot 3:m'$	$\bar{6}'_z, 2_x$	$\{1,2_x,2_{x\sqrt{3}y},2_{\bar{x}\sqrt{3}y},\pm 3_z\},\{m'_y,m'_{\sqrt{3}xy},m'_{\sqrt{3}x\bar{y}},m'_z,\pm\bar{6}'_z\}$	[12]
Hexagonal	$\bar{6}'m2'$	$\bar{6}'m2'$	$\bar{6}'_z m_x 2'_y$	$D_{3h}(C_{3v})$	$m\cdot 3:m'$	$\bar{6}'_z, m_x$	$\{1,m_x,m_{x\sqrt{3}y},m_{\bar{x}\sqrt{3}y},\pm 3_z\},\{2'_y,2'_{\sqrt{3}xy},2'_{\sqrt{3}x\bar{y}},m'_z,\pm\bar{6}'_z\}$	[12]
Hexagonal	$\bar{6}m'2'$	$\bar{6}m'2'$	$\bar{6}_z m'_x 2'_y$	$D_{3h}(C_{3h})$	$m'\cdot 3:m$	$\bar{6}_z, m'_x$	$\{1,m_z,\pm 3_z,\pm\bar{6}_z\},\{2'_y,2'_{\sqrt{3}xy},2'_{\sqrt{3}x\bar{y}},m'_x,m'_{x\sqrt{3}y},m'_{\bar{x}\sqrt{3}y}\}$	[12]
Hexagonal	$6/mmm1'$	$\frac{6}{m}\frac{2}{m}\frac{2}{m}1'$	$6_z/m_z m_x m_y 1'$	D_{6hR}	$m\cdot 6:m1'$	$6_z, m_z, m_x, 1'$	$\{1,\bar{1},2_x,2_{x\sqrt{3}y},2_{\bar{x}\sqrt{3}y},2_y,2_{\sqrt{3}xy},2_{\sqrt{3}x\bar{y}},m_x,m_{x\sqrt{3}y},m_{\bar{x}\sqrt{3}y},$ $m_y,m_{\sqrt{3}xy},m_{\sqrt{3}x\bar{y}},2_z,m_z,\pm 3_z,\pm\bar{3}_z,\pm 6_z,\pm\bar{6}_z\},$ $\{1',\bar{1}',2'_x,2'_{x\sqrt{3}y},2'_{\bar{x}\sqrt{3}y},2'_y,2'_{\sqrt{3}xy},2'_{\sqrt{3}x\bar{y}},m'_x,m'_{x\sqrt{3}y},m'_{\bar{x}\sqrt{3}y},$ $m'_y,m'_{\sqrt{3}xy},m'_{\sqrt{3}x\bar{y}},2'_z,m'_z,\pm 3'_z,\pm\bar{3}'_z,\pm 6'_z,\pm\bar{6}'_z\}$	[48]
Hexagonal	$6/mmm$	$\frac{6}{m}\frac{2}{m}\frac{2}{m}$	$6_z/m_z m_x m_y$	D_{6h}	$m\cdot 6:m$	$6_z, m_z, m_x$	$\{1,\bar{1},2_x,2_{x\sqrt{3}y},2_{\bar{x}\sqrt{3}y},2_y,2_{\sqrt{3}xy},2_{\sqrt{3}x\bar{y}},m_x,m_{x\sqrt{3}y},m_{\bar{x}\sqrt{3}y},$ $m_y,m_{\sqrt{3}xy},m_{\sqrt{3}x\bar{y}},2_z,m_z,\pm 3_z,\pm\bar{3}_z,\pm 6_z,\pm\bar{6}_z\}$	[24]
Hexagonal	$6'/m'mm'$	$\frac{6'}{m'}\frac{2}{m}\frac{2'}{m'}$	$6'_z/m'_z m_x m'_y$	$D_{6h}(D_{3d})$	$m\cdot 6':m'$	$6'_z, m'_z, m_x$	$\{1,\bar{1},2_x,2_{x\sqrt{3}y},2_{\bar{x}\sqrt{3}y},m_x,m_{x\sqrt{3}y},m_{\bar{x}\sqrt{3}y},\pm 3_z,\pm\bar{3}_z\},$ $\{2'_y,2'_{\sqrt{3}xy},2'_{\sqrt{3}x\bar{y}},m'_y,m'_{\sqrt{3}xy},m'_{\sqrt{3}x\bar{y}},2'_z,m'_z,\pm 6'_z,\pm\bar{6}'_z\}$	[24]
Hexagonal	$6/mm'm'$	$\frac{6}{m}\frac{2'}{m'}\frac{2'}{m'}$	$6_z/m_z m'_x m'_y$	$D_{6h}(C_{6h})$	$m'\cdot 6:m$	$6_z, m_z, m'_x$	$\{1,\bar{1},2_z,m_z,\pm 3_z,\pm\bar{3}_z,\pm 6_z,\pm\bar{6}_z\},\{2'_x,2'_{x\sqrt{3}y},2'_{\bar{x}\sqrt{3}y},$ $2'_y,2'_{\sqrt{3}xy},2'_{\sqrt{3}x\bar{y}},m'_x,m'_{x\sqrt{3}y},m'_{\bar{x}\sqrt{3}y},m'_y,m'_{\sqrt{3}xy},m'_{\sqrt{3}x\bar{y}},\}$	[24]
Hexagonal	$6/m'm'm'$	$\frac{6}{m'}\frac{2}{m'}\frac{2}{m'}$	$6_z/m'_z m'_x m'_y$	$D_{6h}(D_6)$	$m'\cdot 6:m'$	$6_z, m'_z, m'_x$	$\{1,2_x,2_{x\sqrt{3}y},2_{\bar{x}\sqrt{3}y},2_y,2_{\sqrt{3}xy},2_{\sqrt{3}x\bar{y}},2_z,\pm 3_z,\pm 6_z\},$ $\{\bar{1}',m'_x,m'_{x\sqrt{3}y},m'_{\bar{x}\sqrt{3}y},m'_y,m'_{\sqrt{3}xy},m'_{\sqrt{3}x\bar{y}},m'_z,\pm\bar{3}'_z,\pm\bar{6}'_z\}$	[24]
Hexagonal	$6/m'mm$	$\frac{6}{m'}\frac{2'}{m}\frac{2'}{m}$	$6_z/m'_z m_x m_y$	$D_{6h}(C_{6v})$	$m\cdot 6:m'$	$6_z, m'_z, m_x$	$\{1,m_x,m_{x\sqrt{3}y},m_{\bar{x}\sqrt{3}y},m_y,m_{\sqrt{3}xy},m_{\sqrt{3}x\bar{y}},2_z,\pm 3_z,\pm 6_z\},$ $\{\bar{1}',2'_x,2'_{x\sqrt{3}y},2'_{\bar{x}\sqrt{3}y},2'_y,2'_{\sqrt{3}xy},2'_{\sqrt{3}x\bar{y}},m'_z,\pm\bar{3}'_z,\pm\bar{6}'_z\}$	[24]
Hexagonal	$6'/mmm'$	$\frac{6'}{m}\frac{2'}{m}\frac{2}{m'}$	$6'_z/m_z m_x m'_y$	$D_{6h}(D_{3h})$	$m\cdot 6':m$	$6'_z, m_z, m_x$	$\{1,2_y,2_{\sqrt{3}xy},2_{\sqrt{3}x\bar{y}},m_x,m_{x\sqrt{3}y},m_{\bar{x}\sqrt{3}y},m_z,\pm 3_z,\pm\bar{6}_z\},$ $\{\bar{1}',2'_x,2'_{x\sqrt{3}y},2'_{\bar{x}\sqrt{3}y},m'_y,m'_{\sqrt{3}xy},m'_{\sqrt{3}x\bar{y}},2'_z,\pm\bar{3}'_z,\pm 6'_z\}$	[24]
		VARIATIONS ON INTERNATIONAL UNION OF CRYTSTALLOGRAPHY						
Hexagonal	$\bar{6}2m$	$\bar{6}2m$	$\bar{6}_z 2_x m_y$	D^x_{3h}	$m\cdot 3:m$	$\bar{6}_z, 2_x$	$\{1,2_x,2_{x\sqrt{3}y},2_{\bar{x}\sqrt{3}y},m_y,m_{\sqrt{3}xy},m_{\sqrt{3}x\bar{y}},m_z,\pm 3_z,\pm\bar{6}_z\}$	[12]



$6mmm1'$

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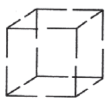
Appendix – Cubic Groups

Family	Point Group Symbol – IUC/Hermann–Mauguin: short	full	annotated	Schoenflies	Shubnikov	Generators	Symmetry Elements	[number]
Cubic	$231'$	$231'$	$2_z 3_{xyz} 1'$	T_R	$3/21'$	$2_z, 3_{xyz}, 1'$	$\{1,\overbrace{2_x,2_y,2_z}^{3(2)},\overbrace{\pm 3_{xyz},\pm 3_{\bar{x}yz},\pm 3_{x\bar{y}z},\pm 3_{xy\bar{z}}}^{4(\pm 3)}\},\{1',3(2'),4(\pm 3')\}$	[24]
Cubic	23	23	$2_z 3_{xyz}$	T	$3/2$	$2_z, 3_{xyz}$	$\{1,3(2),4(\pm 3)\}$	[12]
Cubic	$m\bar{3}1'$	$\frac{2}{m}\bar{3}1'$	$m_z\bar{3}_{xyz}1'$	T_{hR}	$\tilde{6}/21'$	$m_z, \bar{3}_{xyz}, 1'$	$\{1,\bar{1},3(2),\overbrace{m_x,m_y,m_z}^{3(m)},4(\pm 3),\overbrace{\pm\bar{3}_{xyz},\pm\bar{3}_{\bar{x}yz},\pm\bar{3}_{x\bar{y}z},\pm\bar{3}_{xy\bar{z}}}^{4(\pm\bar{3})}\},$ $\{1',\bar{1}',3(2'),3(m'),4(\pm 3'),4(\pm\bar{3}')\}$	[48]
Cubic	$m\bar{3}$	$\frac{2}{m}\bar{3}$	$m_z\bar{3}_{xyz}$	T_h	$\tilde{6}/2$	$m_z, \bar{3}_{xyz}$	$\{1,\bar{1},3(2),3(m),4(\pm 3),4(\pm\bar{3})\}$	[24]
Cubic	$m'\bar{3}'$	$\frac{2}{m'}\bar{3}'$	$m'_z\bar{3}'_{xyz}$	$T_h(T)$	$\tilde{6}'/2$	$m'_z, \bar{3}'_{xyz}$	$\{1,3(2),4(\pm 3)\},\{\bar{1}',3(m'),4(\pm\bar{3}')\}$	[24]
Cubic	$4321'$	$4321'$	$4_z 3_{xyz} 2_{xy} 1'$	O_R	$3/41'$	$4_z, 3_{xyz}, 1'$	$\{1,\overbrace{3(2),\overbrace{2_{xy},2_{x\bar{y}},2_{xz},2_{x\bar{z}},2_{yz},2_{y\bar{z}}}^{6(2)}}^{9(2)},4(\pm 3),\overbrace{\pm 4_x,\pm 4_y,\pm 4_z}^{3(\pm 4)}\},$ $\{1',9(2'),4(\pm 3'),3(\pm 4')\}$	[48]
Cubic	432	432	$4_z 3_{xyz} 2_{xy}$	O	$3/4$	$4_z, 3_{xyz}$	$\{1,9(2),4(\pm 3),3(\pm 4)\}$	[24]
Cubic	$4'32'$	$4'32'$	$4'_z 3_{xyz} 2'_{xy}$	$O(T)$	$3/4'$	$4'_z, 3_{xyz}$	$\{1,3(2),4(\pm 3)\},\{6(2'),3(\pm 4')\}$	[24]
Cubic	$\bar{4}3m1'$	$\bar{4}3m1'$	$\bar{4}_z 3_{xyz} m_{xy} 1'$	T_{dR}	$3/\tilde{4}1'$	$\bar{4}_z, 3_{xyz}, 1'$	$\{1,3(2),\overbrace{m_{xy},m_{x\bar{y}},m_{xz},m_{x\bar{z}},m_{yz},m_{y\bar{z}}}^{6(m)},4(\pm 3),\overbrace{\pm\bar{4}_x,\pm\bar{4}_y,\pm\bar{4}_z}^{3(\pm\bar{4})}\},$ $\{1',3(2'),6(m'),4(\pm 3'),3(\pm\bar{4}')\}$	[48]
Cubic	$\bar{4}3m$	$\bar{4}3m$	$\bar{4}_z 3_{xyz} m_{xy}$	T_d	$3/\tilde{4}$	$\bar{4}_z, 3_{xyz}$	$\{1,3(2),6(m),4(\pm 3),3(\pm\bar{4})\}$	[24]
Cubic	$\bar{4}'3m'$	$\bar{4}'3m'$	$\bar{4}'_z 3_{xyz} m'_{xy}$	$T_d(T)$	$3/\tilde{4}'$	$\bar{4}'_z, 3_{xyz}$	$\{1,3(2),4(\pm 3)\},\{6(m'),3(\pm\bar{4}')\}$	[24]
Cubic	$m\bar{3}m1'$	$\frac{4}{m}\bar{3}\frac{2}{m}1'$	$m_z\bar{3}_{xyz}m_{xy}1'$	O_{hR}	$\tilde{6}/41'$	$4_z, \bar{3}_{xyz}, 1'$	$\{1,\bar{1},9(2),\overbrace{3(m),6(m)}^{9(m)},4(\pm 3),4(\pm\bar{3}),3(\pm 4),3(\pm\bar{4})\},$ $\{1',\bar{1}',9(2'),9(m'),4(\pm 3'),4(\pm\bar{3}'),3(\pm 4'),3(\pm\bar{4}')\}$	[96]
Cubic	$m\bar{3}m$	$\frac{4}{m}\bar{3}\frac{2}{m}$	$m_z\bar{3}_{xyz}m_{xy}$	O_h	$\tilde{6}/4$	$4_z, \bar{3}_{xyz}$	$\{1,\bar{1},9(2),9(m),4(\pm 3),4(\pm\bar{3}),3(\pm 4),3(\pm\bar{4})\}$	[48]
Cubic	$m\bar{3}m'$	$\frac{4'}{m}\bar{3}\frac{2'}{m'}$	$m_z\bar{3}_{xyz}m'_{xy}$	$O_h(T_h)$	$\tilde{6}/4'$	$4'_z, \bar{3}_{xyz}$	$\{1,\bar{1},3(2),3(m),4(\pm 3),4(\pm\bar{3})\},\{6(2'),6(m'),3(\pm 4'),3(\pm\bar{4}')\}$	[48]
Cubic	$m'\bar{3}'m'$	$\frac{4}{m'}\bar{3}'\frac{2}{m'}$	$m'_z\bar{3}'_{xyz}m'_{xy}$	$O_h(O)$	$\tilde{6}'/4$	$4_z, \bar{3}'_{xyz}$	$\{1,9(2),4(\pm 3),3(\pm 4)\},\{\bar{1}',9(m'),4(\pm\bar{3}'),3(\pm\bar{4}')\}$	[48]
Cubic	$m'\bar{3}'m$	$\frac{4'}{m'}\bar{3}'\frac{2'}{m}$	$m'_z\bar{3}'_{xyz}m_{xy}$	$O_h(T_d)$	$\tilde{6}'/4'$	$4'_z, \bar{3}'_{xyz}$	$\{1,3(2),6(m),4(\pm 3),3(\pm\bar{4})\},\{\bar{1}',6(2'),3(m'),4(\pm\bar{3}'),3(\pm 4')\}$	[48]



$m\bar{3}m1'$

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Appendix – Continuous Groups

Point Group Symbol (IUC/Hermann–Mauguin: short, full, annotated; Schoenflies; Shubnikov)

Family	short	full	annotated	Schoenflies	Shubnikov	Generators	Symmetry Elements	[number]
Continuous	$\infty 1'$	$\infty 1'$	$\infty_z 1'$	$C_{\infty R}$	$\infty 1'$	$\infty_z, 1'$	$\{1, \infty_z\}, \{1', \infty'_z\}$	$[\infty]$
Continuous	∞	∞	∞_z	C_∞	∞	∞_z	$\{1, \infty_z\}$	$[\infty]$
Continuous	$\infty/m1'$	$\frac{\infty}{m}1'$	$\infty_z/m_z 1'$	$C_{\infty hR}$	$\infty : m1'$	$\infty_z, m_z, 1'$	$\{1, \bar{1}, m_z, \infty_z\}, \{1', \bar{1}', m'_z, \infty'_z\}^{(a),(e)}$	$[\infty]$
Continuous	∞/m	$\frac{\infty}{m}$	∞_z/m_z	$C_{\infty h}$	$\infty : m$	∞_z, m_z	$\{1, \bar{1}, m_z, \infty_z\}$	$[\infty]$
Continuous	∞/m'	$\frac{\infty}{m'}$	∞_z/m'_z	$C_{\infty h}(C_\infty)$	$\infty : m'$	∞_z, m'_z	$\{1, \infty_z\}, \{\bar{1}', m'_z\}$	$[\infty]$
Continuous	$\infty 21'$	$\infty 21'$	$\infty_z 2_{xy\infty} 1'$	$D_{\infty R}$	$\infty : 21'$	$\infty_z, 2_{xy\infty}, 1'$	$\{1, 2_{xy\infty}, \infty_z\}, \{1', 2'_{xy\infty}, \infty'_z\}$	$[\infty]$
Continuous	$\infty 2$	$\infty 2$	$\infty_z 2_{xy\infty}$	D_∞	$\infty : 2$	$\infty_z, 2_{xy\infty}$	$\{1, 2_{xy\infty}, \infty_z\}$	$[\infty]$
Continuous	$\infty 2'$	$\infty 2'$	$\infty_z 2'_{xy\infty}$	$D_\infty(C_\infty)$	$\infty : 2'$	$\infty_z, 2'_{xy\infty}$	$\{1, \infty_z\}, \{2'_{xy\infty}\}$	$[\infty]$
Continuous	$\infty m1'$	$\infty m1'$	$\infty_z m_{xy\infty} 1'$	$C_{\infty vR}$	$\infty \cdot m1'$	$\infty_z, m_{xy\infty}, 1'$	$\{1, m_{xy\infty}, \infty_z\}, \{1', m'_{xy\infty}, \infty'_z\}$	$[\infty]$
Continuous	∞m	∞m	$\infty_z m_{xy\infty}$	$C_{\infty v}$	$\infty \cdot m$	$\infty_z, m_{xy\infty}$	$\{1, m_{xy\infty}, \infty_z\}$	$[\infty]$
Continuous	$\infty m'$	$\infty m'$	$\infty_z m'_{xy\infty}$	$C_{\infty v}(C_\infty)$	$\infty \cdot m'$	$\infty_z, m'_{xy\infty}$	$\{1, \infty_z\}, \{m'_{xy\infty}\}$	$[\infty]$
Continuous	$\infty/mm1'$	$\frac{\infty}{m}\frac{2}{m}1'$	$\frac{\infty_z}{m_z}\frac{2_{xy\infty}}{m_{xy\infty}}1'$	$D_{\infty hR}$	$m \cdot \infty : m1'$	$\infty_z, m_z, m_{xy\infty}, 1'$	$\{1, \bar{1}, m_z, m_{xy\infty}, 2_{xy\infty}, \infty_z\}, \{1', \bar{1}', m'_z, m'_{xy\infty}, 2'_{xy\infty}, \infty'_z\}^{(b),(h),(i)}$	$[\infty]$
Continuous	∞/mm	$\frac{\infty}{m}\frac{2}{m}$	$\frac{\infty_z}{m_z}\frac{2_{xy\infty}}{m_{xy\infty}}$	$D_{\infty h}$	$m \cdot \infty : m$	$\infty_z, m_z, m_{xy\infty}$	$\{1, \bar{1}, m_z, m_{xy\infty}, 2_{xy\infty}, \infty_z\}$	$[\infty]$
Continuous	∞/mm'	$\frac{\infty}{m}\frac{2'}{m'}$	$\frac{\infty_z}{m_z}\frac{2'_{xy\infty}}{m'_{xy\infty}}$	$D_{\infty h}(C_{\infty h})$	$m' \cdot \infty : m$	$\infty_z, m_z, m'_{xy\infty}$	$\{1, \bar{1}, m_z, \infty_z\}, \{m'_{xy\infty}, 2'_{xy\infty}\}$	$[\infty]$
Continuous	$\infty/m'm$	$\frac{\infty}{m'}\frac{2'}{m}$	$\frac{\infty_z}{m'_z}\frac{2'_{xy\infty}}{m_{xy\infty}}$	$D_{\infty h}(C_{\infty v})$	$m \cdot \infty : m'$	$\infty_z, m'_z, m_{xy\infty}$	$\{1, m_{xy\infty}, \infty_z\}, \{\bar{1}', 2'_{xy\infty}, m'_z\}$	$[\infty]$
Continuous	$\infty/m'm'$	$\frac{\infty}{m'}\frac{2}{m'}$	$\frac{\infty_z}{m'_z}\frac{2_{xy\infty}}{m'_{xy\infty}}$	$D_{\infty h}(D_\infty)$	$m' \cdot \infty : m'$	$\infty_z, m'_z, m'_{xy\infty}$	$\{1, 2_{xy\infty}, \infty_z\}, \{\bar{1}', m'_z, m'_{xy\infty}\}$	$[\infty]$
Continuous	$\infty\infty 1'$	$\infty\infty 1'$	$\infty_\infty 1'$	K_R	$\infty/\infty 1'$	$\infty_\infty, 1'$	$\{1, \infty_\infty\}, \{1', \infty'_\infty\}^{(c)}$	$[\infty]$
Continuous	$\infty\infty$	$\infty\infty$	∞_∞	K	∞/∞	∞_∞	$\{1, \infty_\infty\}$	$[\infty]$
Continuous	$\infty\infty m1'$	$\infty\infty m1'$	$\infty_\infty m_\infty 1'$	K_{hR}	$\infty/\infty \cdot m1'$	$\infty_\infty, m_\infty, 1'$	$\{1, \bar{1}, m_\infty, \infty_\infty\}, \{1', \bar{1}', m'_\infty, \infty'_\infty\}^{(d),(f),(g)}$	$[\infty]$
Continuous	$\infty\infty m$	$\infty\infty m$	$\infty_\infty m_\infty$	K_h	$\infty/\infty \cdot m$	∞_∞, m_∞	$\{1, \bar{1}, m_\infty, \infty_\infty\}$	$[\infty]$
Continuous	$\infty\infty m'$	$\infty\infty m'$	$\infty_\infty m'_\infty$	$K_h(K)$	$\infty/\infty \cdot m'$	∞_∞, m'_∞	$\{1, \infty_\infty\}, \{\bar{1}', m'_\infty\}$	$[\infty]$

NOTES:

(a) $2_z \subset \infty_z$ and $m_z 2_z = \bar{1}$

(b) $m_{xy\infty} \bar{1} = 2_{xy\infty}$

(c) $\infty_x, \infty_y, \infty_z \subset \infty_\infty$

(d) $m_x, m_y, m_z \subset m_\infty$

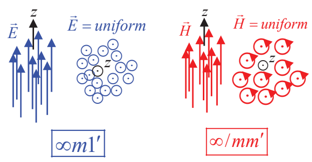
(e) ∞_z = Continuous rotation symmetry about z-axis

(f) ∞_∞ = Infinite number of continuous axes of symmetry

(g) m_∞ = Infinite number of mirror planes of symmetry

(h) $m_{xy\infty}$ = Infinite number of mirror planes passing through z-axis

(i) $2_{xy\infty}$ = Infinite number of two-fold axes lying in the x-y plane



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Appendix – Material Property Tensors (Triclinic)

Material Tensors

Family	Group	$\bar{\varepsilon}$	$\bar{\mu}$	$\bar{\xi}$	$\bar{\zeta}$	[# of parameters]
Triclinic	$11'$	$\begin{bmatrix}\varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz}\\ \varepsilon_{xy} & \varepsilon_{yy} & \varepsilon_{yz}\\ \varepsilon_{xz} & \varepsilon_{yz} & \varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx} & \mu_{xy} & \mu_{xz}\\ \mu_{xy} & \mu_{yy} & \mu_{yz}\\ \mu_{xz} & \mu_{yz} & \mu_{zz}\end{bmatrix}$	$\begin{bmatrix}\xi_{xx} & \xi_{xy} & \xi_{xz}\\ \xi_{yx} & \xi_{yy} & \xi_{yz}\\ \xi_{zx} & \xi_{zy} & \xi_{zz}\end{bmatrix}$	$\begin{bmatrix}-\xi_{xx} & -\xi_{yx} & -\xi_{zx}\\ -\xi_{xy} & -\xi_{yy} & -\xi_{zy}\\ -\xi_{xz} & -\xi_{yz} & -\xi_{zz}\end{bmatrix}$	[21]
Triclinic	(1)	$\begin{bmatrix}\varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz}\\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz}\\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx} & \mu_{xy} & \mu_{xz}\\ \mu_{yx} & \mu_{yy} & \mu_{yz}\\ \mu_{zx} & \mu_{zy} & \mu_{zz}\end{bmatrix}$	$\begin{bmatrix}\xi_{xx} & \xi_{xy} & \xi_{xz}\\ \xi_{yx} & \xi_{yy} & \xi_{yz}\\ \xi_{zx} & \xi_{zy} & \xi_{zz}\end{bmatrix}$	$\begin{bmatrix}\zeta_{xx} & \zeta_{xy} & \zeta_{xz}\\ \zeta_{yx} & \zeta_{yy} & \zeta_{yz}\\ \zeta_{zx} & \zeta_{zy} & \zeta_{zz}\end{bmatrix}$	[36]
Triclinic	$\bar{1}1'$	$\begin{bmatrix}\varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz}\\ \varepsilon_{xy} & \varepsilon_{yy} & \varepsilon_{yz}\\ \varepsilon_{xz} & \varepsilon_{yz} & \varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx} & \mu_{xy} & \mu_{xz}\\ \mu_{xy} & \mu_{yy} & \mu_{yz}\\ \mu_{xz} & \mu_{yz} & \mu_{zz}\end{bmatrix}$	$\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$	$\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$	[12]
Triclinic	($\bar{1}$)	$\begin{bmatrix}\varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz}\\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz}\\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx} & \mu_{xy} & \mu_{xz}\\ \mu_{yx} & \mu_{yy} & \mu_{yz}\\ \mu_{zx} & \mu_{zy} & \mu_{zz}\end{bmatrix}$	$\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$	$\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$	[18]
Triclinic	($\bar{1}'$)	$\begin{bmatrix}\varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz}\\ \varepsilon_{xy} & \varepsilon_{yy} & \varepsilon_{yz}\\ \varepsilon_{xz} & \varepsilon_{yz} & \varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx} & \mu_{xy} & \mu_{xz}\\ \mu_{xy} & \mu_{yy} & \mu_{yz}\\ \mu_{xz} & \mu_{yz} & \mu_{zz}\end{bmatrix}$	$\begin{bmatrix}\xi_{xx} & \xi_{xy} & \xi_{xz}\\ \xi_{yx} & \xi_{yy} & \xi_{yz}\\ \xi_{zx} & \xi_{zy} & \xi_{zz}\end{bmatrix}$	$\begin{bmatrix}\xi_{xx} & \xi_{yx} & \xi_{zx}\\ \xi_{xy} & \xi_{yy} & \xi_{zy}\\ \xi_{xz} & \xi_{yz} & \xi_{zz}\end{bmatrix}$	[21]

( ) Not Reciprocal!

$$RECIPROCAL \Rightarrow \bar{\varepsilon} = \bar{\varepsilon}^T \ , \ \bar{\mu} = \bar{\mu}^T \ , \ \bar{\zeta} = -\bar{\xi}^T$$

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Appendix – Material Property Tensors (Monoclinic)

Family	*Group*	*Material Tensors* $\overleftrightarrow{\varepsilon}$	$\overleftrightarrow{\mu}$	$\overleftrightarrow{\xi}$	$\overleftrightarrow{\zeta}$	[# of parameters]
Monoclinic	$21'$	$\begin{bmatrix}\varepsilon_{xx}&0&\varepsilon_{xz}\\0&\varepsilon_{yy}&0\\\varepsilon_{xz}&0&\varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx}&0&\mu_{xz}\\0&\mu_{yy}&0\\\mu_{xz}&0&\mu_{zz}\end{bmatrix}$	$\begin{bmatrix}\xi_{xx}&0&\xi_{xz}\\0&\xi_{yy}&0\\\xi_{zx}&0&\xi_{zz}\end{bmatrix}$	$\begin{bmatrix}-\xi_{xx}&0&-\xi_{zx}\\0&-\xi_{yy}&0\\-\xi_{xz}&0&-\xi_{zz}\end{bmatrix}$	[13]
Monoclinic	2	$\begin{bmatrix}\varepsilon_{xx}&0&\varepsilon_{xz}\\0&\varepsilon_{yy}&0\\\varepsilon_{zx}&0&\varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx}&0&\mu_{xz}\\0&\mu_{yy}&0\\\mu_{zx}&0&\mu_{zz}\end{bmatrix}$	$\begin{bmatrix}\xi_{xx}&0&\xi_{xz}\\0&\xi_{yy}&0\\\xi_{zx}&0&\xi_{zz}\end{bmatrix}$	$\begin{bmatrix}\zeta_{xx}&0&\zeta_{xz}\\0&\zeta_{yy}&0\\\zeta_{zx}&0&\zeta_{zz}\end{bmatrix}$	[20]
Monoclinic	$2'$	$\begin{bmatrix}\varepsilon_{xx}&\varepsilon_{xy}&\varepsilon_{xz}\\-\varepsilon_{xy}&\varepsilon_{yy}&\varepsilon_{yz}\\\varepsilon_{xz}&-\varepsilon_{yz}&\varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx}&\mu_{xy}&\mu_{xz}\\-\mu_{xy}&\mu_{yy}&\mu_{yz}\\\mu_{xz}&-\mu_{yz}&\mu_{zz}\end{bmatrix}$	$\begin{bmatrix}\xi_{xx}&\xi_{xy}&\xi_{xz}\\\xi_{yx}&\xi_{yy}&\xi_{yz}\\\xi_{zx}&\xi_{zy}&\xi_{zz}\end{bmatrix}$	$\begin{bmatrix}-\xi_{xx}&\xi_{yx}&-\xi_{zx}\\\xi_{xy}&-\xi_{yy}&\xi_{zy}\\-\xi_{xz}&\xi_{yz}&-\xi_{zz}\end{bmatrix}$	[21]
Monoclinic	$m1'$	$\begin{bmatrix}\varepsilon_{xx}&0&\varepsilon_{xz}\\0&\varepsilon_{yy}&0\\\varepsilon_{xz}&0&\varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx}&0&\mu_{xz}\\0&\mu_{yy}&0\\\mu_{xz}&0&\mu_{zz}\end{bmatrix}$	$\begin{bmatrix}0&\xi_{xy}&0\\\xi_{yx}&0&\xi_{yz}\\0&\xi_{zy}&0\end{bmatrix}$	$\begin{bmatrix}0&-\xi_{yx}&0\\-\xi_{xy}&0&-\xi_{zy}\\0&-\xi_{yz}&0\end{bmatrix}$	[12]
Monoclinic	m	$\begin{bmatrix}\varepsilon_{xx}&0&\varepsilon_{xz}\\0&\varepsilon_{yy}&0\\\varepsilon_{zx}&0&\varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx}&0&\mu_{xz}\\0&\mu_{yy}&0\\\mu_{zx}&0&\mu_{zz}\end{bmatrix}$	$\begin{bmatrix}0&\xi_{xy}&0\\\xi_{yx}&0&\xi_{yz}\\0&\xi_{zy}&0\end{bmatrix}$	$\begin{bmatrix}0&\zeta_{xy}&0\\\zeta_{yx}&0&\zeta_{yz}\\0&\zeta_{zy}&0\end{bmatrix}$	[18]
Monoclinic	m'	$\begin{bmatrix}\varepsilon_{xx}&\varepsilon_{xy}&\varepsilon_{xz}\\-\varepsilon_{xy}&\varepsilon_{yy}&\varepsilon_{yz}\\\varepsilon_{xz}&-\varepsilon_{yz}&\varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx}&\mu_{xy}&\mu_{xz}\\-\mu_{xy}&\mu_{yy}&\mu_{yz}\\\mu_{xz}&-\mu_{yz}&\mu_{zz}\end{bmatrix}$	$\begin{bmatrix}\xi_{xx}&\xi_{xy}&\xi_{xz}\\\xi_{yx}&\xi_{yy}&\xi_{yz}\\\xi_{zx}&\xi_{zy}&\xi_{zz}\end{bmatrix}$	$\begin{bmatrix}\xi_{xx}&-\xi_{yx}&\xi_{zx}\\-\xi_{xy}&\xi_{yy}&-\xi_{zy}\\\xi_{xz}&-\xi_{yz}&\xi_{zz}\end{bmatrix}$	[21]
Monoclinic	$2/m1'$	$\begin{bmatrix}\varepsilon_{xx}&0&\varepsilon_{xz}\\0&\varepsilon_{yy}&0\\\varepsilon_{xz}&0&\varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx}&0&\mu_{xz}\\0&\mu_{yy}&0\\\mu_{xz}&0&\mu_{zz}\end{bmatrix}$	$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	[8]
Monoclinic	$2/m$	$\begin{bmatrix}\varepsilon_{xx}&0&\varepsilon_{xz}\\0&\varepsilon_{yy}&0\\\varepsilon_{zx}&0&\varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx}&0&\mu_{xz}\\0&\mu_{yy}&0\\\mu_{zx}&0&\mu_{zz}\end{bmatrix}$	$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	[10]
Monoclinic	$2'/m'$	$\begin{bmatrix}\varepsilon_{xx}&\varepsilon_{xy}&\varepsilon_{xz}\\-\varepsilon_{xy}&\varepsilon_{yy}&\varepsilon_{yz}\\\varepsilon_{xz}&-\varepsilon_{yz}&\varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx}&\mu_{xy}&\mu_{xz}\\-\mu_{xy}&\mu_{yy}&\mu_{yz}\\\mu_{xz}&-\mu_{yz}&\mu_{zz}\end{bmatrix}$	$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	[12]
Monoclinic	$2/m'$	$\begin{bmatrix}\varepsilon_{xx}&0&\varepsilon_{xz}\\0&\varepsilon_{yy}&0\\\varepsilon_{xz}&0&\varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx}&0&\mu_{xz}\\0&\mu_{yy}&0\\\mu_{xz}&0&\mu_{zz}\end{bmatrix}$	$\begin{bmatrix}\xi_{xx}&0&\xi_{xz}\\0&\xi_{yy}&0\\\xi_{zx}&0&\xi_{zz}\end{bmatrix}$	$\begin{bmatrix}\xi_{xx}&0&\xi_{zx}\\0&\xi_{yy}&0\\\xi_{xz}&0&\xi_{zz}\end{bmatrix}$	[13]
Monoclinic	$2'/m$	$\begin{bmatrix}\varepsilon_{xx}&0&\varepsilon_{xz}\\0&\varepsilon_{yy}&0\\\varepsilon_{xz}&0&\varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx}&0&\mu_{xz}\\0&\mu_{yy}&0\\\mu_{xz}&0&\mu_{zz}\end{bmatrix}$	$\begin{bmatrix}0&\xi_{xy}&0\\\xi_{yx}&0&\xi_{yz}\\0&\xi_{zy}&0\end{bmatrix}$	$\begin{bmatrix}0&\xi_{yx}&0\\\xi_{xy}&0&\xi_{zy}\\0&\xi_{yz}&0\end{bmatrix}$	[12]

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Appendix – Material Property Tensors (Orthorhombic)

Family	*Group*	*Material Tensors* $\overleftrightarrow{\varepsilon}$	$\overleftrightarrow{\mu}$	$\overleftrightarrow{\xi}$	$\overleftrightarrow{\zeta}$	[# of parameters]
Orthorhombic	$2221'$	$\begin{bmatrix}\varepsilon_{xx}&0&0\\0&\varepsilon_{yy}&0\\0&0&\varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx}&0&0\\0&\mu_{yy}&0\\0&0&\mu_{zz}\end{bmatrix}$	$\begin{bmatrix}\xi_{xx}&0&0\\0&\xi_{yy}&0\\0&0&\xi_{zz}\end{bmatrix}$	$\begin{bmatrix}-\xi_{xx}&0&0\\0&-\xi_{yy}&0\\0&0&-\xi_{zz}\end{bmatrix}$	[9]
Orthorhombic	222	$\begin{bmatrix}\varepsilon_{xx}&0&0\\0&\varepsilon_{yy}&0\\0&0&\varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx}&0&0\\0&\mu_{yy}&0\\0&0&\mu_{zz}\end{bmatrix}$	$\begin{bmatrix}\xi_{xx}&0&0\\0&\xi_{yy}&0\\0&0&\xi_{zz}\end{bmatrix}$	$\begin{bmatrix}\zeta_{xx}&0&0\\0&\zeta_{yy}&0\\0&0&\zeta_{zz}\end{bmatrix}$	[12]
Orthorhombic	$2'2'2$	$\begin{bmatrix}\varepsilon_{xx}&\varepsilon_{xy}&0\\-\varepsilon_{xy}&\varepsilon_{yy}&0\\0&0&\varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx}&\mu_{xy}&0\\-\mu_{xy}&\mu_{yy}&0\\0&0&\mu_{zz}\end{bmatrix}$	$\begin{bmatrix}\xi_{xx}&\xi_{xy}&0\\\xi_{yx}&\xi_{yy}&0\\0&0&\xi_{zz}\end{bmatrix}$	$\begin{bmatrix}-\xi_{xx}&\xi_{yx}&0\\\xi_{xy}&-\xi_{yy}&0\\0&0&-\xi_{zz}\end{bmatrix}$	[13]
Orthorhombic	$mm21'$	$\begin{bmatrix}\varepsilon_{xx}&0&0\\0&\varepsilon_{yy}&0\\0&0&\varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx}&0&0\\0&\mu_{yy}&0\\0&0&\mu_{zz}\end{bmatrix}$	$\begin{bmatrix}0&\xi_{xy}&0\\\xi_{yx}&0&0\\0&0&0\end{bmatrix}$	$\begin{bmatrix}0&-\xi_{yx}&0\\-\xi_{xy}&0&0\\0&0&0\end{bmatrix}$	[8]
Orthorhombic	$mm2$	$\begin{bmatrix}\varepsilon_{xx}&0&0\\0&\varepsilon_{yy}&0\\0&0&\varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx}&0&0\\0&\mu_{yy}&0\\0&0&\mu_{zz}\end{bmatrix}$	$\begin{bmatrix}0&\xi_{xy}&0\\\xi_{yx}&0&0\\0&0&0\end{bmatrix}$	$\begin{bmatrix}0&\zeta_{xy}&0\\\zeta_{yx}&0&0\\0&0&0\end{bmatrix}$	[10]
Orthorhombic	$m'm'2$	$\begin{bmatrix}\varepsilon_{xx}&\varepsilon_{xy}&0\\-\varepsilon_{xy}&\varepsilon_{yy}&0\\0&0&\varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx}&\mu_{xy}&0\\-\mu_{xy}&\mu_{yy}&0\\0&0&\mu_{zz}\end{bmatrix}$	$\begin{bmatrix}\xi_{xx}&\xi_{xy}&0\\\xi_{yx}&\xi_{yy}&0\\0&0&\xi_{zz}\end{bmatrix}$	$\begin{bmatrix}\xi_{xx}&-\xi_{yx}&0\\-\xi_{xy}&\xi_{yy}&0\\0&0&\xi_{zz}\end{bmatrix}$	[13]
Orthorhombic	$2'm'm$	$\begin{bmatrix}\varepsilon_{xx}&\varepsilon_{xy}&0\\-\varepsilon_{xy}&\varepsilon_{yy}&0\\0&0&\varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx}&\mu_{xy}&0\\-\mu_{xy}&\mu_{yy}&0\\0&0&\mu_{zz}\end{bmatrix}$	$\begin{bmatrix}0&0&\xi_{xz}\\0&0&\xi_{yz}\\\xi_{zx}&\xi_{zy}&0\end{bmatrix}$	$\begin{bmatrix}0&0&\xi_{zx}\\0&0&-\xi_{zy}\\\xi_{xz}&-\xi_{yz}&0\end{bmatrix}$	[12]
Orthorhombic	$mmm1'$	$\begin{bmatrix}\varepsilon_{xx}&0&0\\0&\varepsilon_{yy}&0\\0&0&\varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx}&0&0\\0&\mu_{yy}&0\\0&0&\mu_{zz}\end{bmatrix}$	$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	[6]
Orthorhombic	mmm	$\begin{bmatrix}\varepsilon_{xx}&0&0\\0&\varepsilon_{yy}&0\\0&0&\varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx}&0&0\\0&\mu_{yy}&0\\0&0&\mu_{zz}\end{bmatrix}$	$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	[6]
Orthorhombic	$mm'm'$	$\begin{bmatrix}\varepsilon_{xx}&0&0\\0&\varepsilon_{yy}&\varepsilon_{yz}\\0&-\varepsilon_{yz}&\varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx}&0&0\\0&\mu_{yy}&\mu_{yz}\\0&-\mu_{yz}&\mu_{zz}\end{bmatrix}$	$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	[8]
Orthorhombic	$m'm'm'$	$\begin{bmatrix}\varepsilon_{xx}&0&0\\0&\varepsilon_{yy}&0\\0&0&\varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx}&0&0\\0&\mu_{yy}&0\\0&0&\mu_{zz}\end{bmatrix}$	$\begin{bmatrix}\xi_{xx}&0&0\\0&\xi_{yy}&0\\0&0&\xi_{zz}\end{bmatrix}$	$\begin{bmatrix}\xi_{xx}&0&0\\0&\xi_{yy}&0\\0&0&\xi_{zz}\end{bmatrix}$	[9]
Orthorhombic	mmm'	$\begin{bmatrix}\varepsilon_{xx}&0&0\\0&\varepsilon_{yy}&0\\0&0&\varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx}&0&0\\0&\mu_{yy}&0\\0&0&\mu_{zz}\end{bmatrix}$	$\begin{bmatrix}0&\xi_{xy}&0\\\xi_{yx}&0&0\\0&0&0\end{bmatrix}$	$\begin{bmatrix}0&\xi_{yx}&0\\\xi_{xy}&0&0\\0&0&0\end{bmatrix}$	[8]

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Appendix – Material Property Tensors (Tetragonal-1)

Family	*Group*	*Material Tensors* $\bar{\bar{\varepsilon}}$	$\bar{\bar{\mu}}$	$\bar{\bar{\xi}}$	$\bar{\bar{\zeta}}$	[*# of parameters*]
Tetragonal	$41'$	$\begin{bmatrix}\varepsilon_{xx} & 0 & 0\\ 0 & \varepsilon_{xx} & 0\\ 0 & 0 & \varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx} & 0 & 0\\ 0 & \mu_{xx} & 0\\ 0 & 0 & \mu_{zz}\end{bmatrix}$	$\begin{bmatrix}\xi_{xx} & \xi_{xy} & 0\\ -\xi_{xy} & \xi_{xx} & 0\\ 0 & 0 & \xi_{zz}\end{bmatrix}$	$\begin{bmatrix}-\xi_{xx} & \xi_{xy} & 0\\ -\xi_{xy} & -\xi_{xx} & 0\\ 0 & 0 & -\xi_{zz}\end{bmatrix}$	[7]
Tetragonal	4	$\begin{bmatrix}\varepsilon_{xx} & \varepsilon_{xy} & 0\\ -\varepsilon_{xy} & \varepsilon_{xx} & 0\\ 0 & 0 & \varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx} & \mu_{xy} & 0\\ -\mu_{xy} & \mu_{xx} & 0\\ 0 & 0 & \mu_{zz}\end{bmatrix}$	$\begin{bmatrix}\xi_{xx} & \xi_{xy} & 0\\ -\xi_{xy} & \xi_{xx} & 0\\ 0 & 0 & \xi_{zz}\end{bmatrix}$	$\begin{bmatrix}\zeta_{xx} & \zeta_{xy} & 0\\ -\zeta_{xy} & \zeta_{xx} & 0\\ 0 & 0 & \zeta_{zz}\end{bmatrix}$	[12]
Tetragonal	$4'$	$\begin{bmatrix}\varepsilon_{xx} & 0 & 0\\ 0 & \varepsilon_{xx} & 0\\ 0 & 0 & \varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx} & 0 & 0\\ 0 & \mu_{xx} & 0\\ 0 & 0 & \mu_{zz}\end{bmatrix}$	$\begin{bmatrix}\xi_{xx} & \xi_{xy} & 0\\ \xi_{yx} & \xi_{yy} & 0\\ 0 & 0 & \xi_{zz}\end{bmatrix}$	$\begin{bmatrix}-\xi_{yy} & \xi_{xy} & 0\\ \xi_{yx} & -\xi_{xx} & 0\\ 0 & 0 & -\xi_{zz}\end{bmatrix}$	[9]
Tetragonal	$\bar{4}1'$	$\begin{bmatrix}\varepsilon_{xx} & 0 & 0\\ 0 & \varepsilon_{xx} & 0\\ 0 & 0 & \varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx} & 0 & 0\\ 0 & \mu_{xx} & 0\\ 0 & 0 & \mu_{zz}\end{bmatrix}$	$\begin{bmatrix}\xi_{xx} & \xi_{xy} & 0\\ \xi_{xy} & -\xi_{xx} & 0\\ 0 & 0 & 0\end{bmatrix}$	$\begin{bmatrix}-\xi_{xx} & -\xi_{xy} & 0\\ -\xi_{xy} & \xi_{xx} & 0\\ 0 & 0 & 0\end{bmatrix}$	[6]
Tetragonal	$\bar{4}$	$\begin{bmatrix}\varepsilon_{xx} & \varepsilon_{xy} & 0\\ -\varepsilon_{xy} & \varepsilon_{xx} & 0\\ 0 & 0 & \varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx} & \mu_{xy} & 0\\ -\mu_{xy} & \mu_{xx} & 0\\ 0 & 0 & \mu_{zz}\end{bmatrix}$	$\begin{bmatrix}\xi_{xx} & \xi_{xy} & 0\\ \xi_{xy} & -\xi_{xx} & 0\\ 0 & 0 & 0\end{bmatrix}$	$\begin{bmatrix}\zeta_{xx} & \zeta_{xy} & 0\\ \zeta_{xy} & -\zeta_{xx} & 0\\ 0 & 0 & 0\end{bmatrix}$	[10]
Tetragonal	$\bar{4}'$	$\begin{bmatrix}\varepsilon_{xx} & 0 & 0\\ 0 & \varepsilon_{xx} & 0\\ 0 & 0 & \varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx} & 0 & 0\\ 0 & \mu_{xx} & 0\\ 0 & 0 & \mu_{zz}\end{bmatrix}$	$\begin{bmatrix}\xi_{xx} & \xi_{xy} & 0\\ \xi_{yx} & \xi_{yy} & 0\\ 0 & 0 & \xi_{zz}\end{bmatrix}$	$\begin{bmatrix}\xi_{yy} & -\xi_{xy} & 0\\ -\xi_{yx} & \xi_{xx} & 0\\ 0 & 0 & \xi_{zz}\end{bmatrix}$	[9]
Tetragonal	$4/m1'$	$\begin{bmatrix}\varepsilon_{xx} & 0 & 0\\ 0 & \varepsilon_{xx} & 0\\ 0 & 0 & \varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx} & 0 & 0\\ 0 & \mu_{xx} & 0\\ 0 & 0 & \mu_{zz}\end{bmatrix}$	$\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$	$\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$	[4]
Tetragonal	$4/m$	$\begin{bmatrix}\varepsilon_{xx} & \varepsilon_{xy} & 0\\ -\varepsilon_{xy} & \varepsilon_{xx} & 0\\ 0 & 0 & \varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx} & \mu_{xy} & 0\\ -\mu_{xy} & \mu_{xx} & 0\\ 0 & 0 & \mu_{zz}\end{bmatrix}$	$\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$	$\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$	[6]
Tetragonal	$4'/m$	$\begin{bmatrix}\varepsilon_{xx} & 0 & 0\\ 0 & \varepsilon_{xx} & 0\\ 0 & 0 & \varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx} & 0 & 0\\ 0 & \mu_{xx} & 0\\ 0 & 0 & \mu_{zz}\end{bmatrix}$	$\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$	$\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$	[4]
Tetragonal	$4/m'$	$\begin{bmatrix}\varepsilon_{xx} & 0 & 0\\ 0 & \varepsilon_{xx} & 0\\ 0 & 0 & \varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx} & 0 & 0\\ 0 & \mu_{xx} & 0\\ 0 & 0 & \mu_{zz}\end{bmatrix}$	$\begin{bmatrix}\xi_{xx} & \xi_{xy} & 0\\ -\xi_{xy} & \xi_{xx} & 0\\ 0 & 0 & \xi_{zz}\end{bmatrix}$	$\begin{bmatrix}\xi_{xx} & -\xi_{xy} & 0\\ \xi_{xy} & \xi_{xx} & 0\\ 0 & 0 & \xi_{zz}\end{bmatrix}$	[7]
Tetragonal	$4'/m'$	$\begin{bmatrix}\varepsilon_{xx} & 0 & 0\\ 0 & \varepsilon_{xx} & 0\\ 0 & 0 & \varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx} & 0 & 0\\ 0 & \mu_{xx} & 0\\ 0 & 0 & \mu_{zz}\end{bmatrix}$	$\begin{bmatrix}\xi_{xx} & \xi_{xy} & 0\\ \xi_{xy} & -\xi_{xx} & 0\\ 0 & 0 & 0\end{bmatrix}$	$\begin{bmatrix}\xi_{xx} & \xi_{xy} & 0\\ \xi_{xy} & -\xi_{xx} & 0\\ 0 & 0 & 0\end{bmatrix}$	[6]

Appendix – Material Property Tensors (Tetragonal-2)

Family	*Group*	*Material Tensors* $\bar{\bar{\varepsilon}}$	$\bar{\bar{\mu}}$	$\bar{\bar{\xi}}$	$\bar{\bar{\zeta}}$	[*# of parameters*]
Tetragonal	$4221'$	$\begin{bmatrix}\varepsilon_{xx} & 0 & 0\\ 0 & \varepsilon_{xx} & 0\\ 0 & 0 & \varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx} & 0 & 0\\ 0 & \mu_{xx} & 0\\ 0 & 0 & \mu_{zz}\end{bmatrix}$	$\begin{bmatrix}\xi_{xx} & 0 & 0\\ 0 & \xi_{xx} & 0\\ 0 & 0 & \xi_{zz}\end{bmatrix}$	$\begin{bmatrix}-\xi_{xx} & 0 & 0\\ 0 & -\xi_{xx} & 0\\ 0 & 0 & -\xi_{zz}\end{bmatrix}$	[6]
Tetragonal	422	$\begin{bmatrix}\varepsilon_{xx} & 0 & 0\\ 0 & \varepsilon_{xx} & 0\\ 0 & 0 & \varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx} & 0 & 0\\ 0 & \mu_{xx} & 0\\ 0 & 0 & \mu_{zz}\end{bmatrix}$	$\begin{bmatrix}\xi_{xx} & 0 & 0\\ 0 & \xi_{xx} & 0\\ 0 & 0 & \xi_{zz}\end{bmatrix}$	$\begin{bmatrix}\zeta_{xx} & 0 & 0\\ 0 & \zeta_{xx} & 0\\ 0 & 0 & \zeta_{zz}\end{bmatrix}$	[8]
Tetragonal	$4'22'$	$\begin{bmatrix}\varepsilon_{xx} & 0 & 0\\ 0 & \varepsilon_{xx} & 0\\ 0 & 0 & \varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx} & 0 & 0\\ 0 & \mu_{xx} & 0\\ 0 & 0 & \mu_{zz}\end{bmatrix}$	$\begin{bmatrix}\xi_{xx} & 0 & 0\\ 0 & \xi_{yy} & 0\\ 0 & 0 & \xi_{zz}\end{bmatrix}$	$\begin{bmatrix}-\xi_{yy} & 0 & 0\\ 0 & -\xi_{xx} & 0\\ 0 & 0 & -\xi_{zz}\end{bmatrix}$	[7]
Tetragonal	$42'2'$	$\begin{bmatrix}\varepsilon_{xx} & \varepsilon_{xy} & 0\\ -\varepsilon_{xy} & \varepsilon_{xx} & 0\\ 0 & 0 & \varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx} & \mu_{xy} & 0\\ -\mu_{xy} & \mu_{xx} & 0\\ 0 & 0 & \mu_{zz}\end{bmatrix}$	$\begin{bmatrix}\xi_{xx} & \xi_{xy} & 0\\ -\xi_{xy} & \xi_{xx} & 0\\ 0 & 0 & \xi_{zz}\end{bmatrix}$	$\begin{bmatrix}-\xi_{xx} & -\xi_{xy} & 0\\ \xi_{xy} & -\xi_{xx} & 0\\ 0 & 0 & -\xi_{zz}\end{bmatrix}$	[9]
Tetragonal	$4mm1'$	$\begin{bmatrix}\varepsilon_{xx} & 0 & 0\\ 0 & \varepsilon_{xx} & 0\\ 0 & 0 & \varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx} & 0 & 0\\ 0 & \mu_{xx} & 0\\ 0 & 0 & \mu_{zz}\end{bmatrix}$	$\begin{bmatrix}0 & \xi_{xy} & 0\\ -\xi_{xy} & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$	$\begin{bmatrix}0 & \xi_{xy} & 0\\ -\xi_{xy} & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$	[5]
Tetragonal	$4mm$	$\begin{bmatrix}\varepsilon_{xx} & 0 & 0\\ 0 & \varepsilon_{xx} & 0\\ 0 & 0 & \varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx} & 0 & 0\\ 0 & \mu_{xx} & 0\\ 0 & 0 & \mu_{zz}\end{bmatrix}$	$\begin{bmatrix}0 & \xi_{xy} & 0\\ -\xi_{xy} & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$	$\begin{bmatrix}0 & \zeta_{xy} & 0\\ -\zeta_{xy} & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$	[6]
Tetragonal	$4'mm'$	$\begin{bmatrix}\varepsilon_{xx} & 0 & 0\\ 0 & \varepsilon_{xx} & 0\\ 0 & 0 & \varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx} & 0 & 0\\ 0 & \mu_{xx} & 0\\ 0 & 0 & \mu_{zz}\end{bmatrix}$	$\begin{bmatrix}0 & \xi_{xy} & 0\\ \xi_{yx} & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$	$\begin{bmatrix}0 & \xi_{xy} & 0\\ \xi_{yx} & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$	[6]
Tetragonal	$4m'm'$	$\begin{bmatrix}\varepsilon_{xx} & \varepsilon_{xy} & 0\\ -\varepsilon_{xy} & \varepsilon_{xx} & 0\\ 0 & 0 & \varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx} & \mu_{xy} & 0\\ -\mu_{xy} & \mu_{xx} & 0\\ 0 & 0 & \mu_{zz}\end{bmatrix}$	$\begin{bmatrix}\xi_{xx} & \xi_{xy} & 0\\ -\xi_{xy} & \xi_{xx} & 0\\ 0 & 0 & \xi_{zz}\end{bmatrix}$	$\begin{bmatrix}\xi_{xx} & \xi_{xy} & 0\\ -\xi_{xy} & \xi_{xx} & 0\\ 0 & 0 & \xi_{zz}\end{bmatrix}$	[9]
Tetragonal	$\bar{4}2m1'$	$\begin{bmatrix}\varepsilon_{xx} & 0 & 0\\ 0 & \varepsilon_{xx} & 0\\ 0 & 0 & \varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx} & 0 & 0\\ 0 & \mu_{xx} & 0\\ 0 & 0 & \mu_{zz}\end{bmatrix}$	$\begin{bmatrix}\xi_{xx} & 0 & 0\\ 0 & -\xi_{xx} & 0\\ 0 & 0 & 0\end{bmatrix}$	$\begin{bmatrix}-\xi_{xx} & 0 & 0\\ 0 & \xi_{xx} & 0\\ 0 & 0 & 0\end{bmatrix}$	[5]
Tetragonal	$\bar{4}2m$	$\begin{bmatrix}\varepsilon_{xx} & 0 & 0\\ 0 & \varepsilon_{xx} & 0\\ 0 & 0 & \varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx} & 0 & 0\\ 0 & \mu_{xx} & 0\\ 0 & 0 & \mu_{zz}\end{bmatrix}$	$\begin{bmatrix}\xi_{xx} & 0 & 0\\ 0 & -\xi_{xx} & 0\\ 0 & 0 & 0\end{bmatrix}$	$\begin{bmatrix}\zeta_{xx} & 0 & 0\\ 0 & -\zeta_{xx} & 0\\ 0 & 0 & 0\end{bmatrix}$	[6]

Appendix – Material Property Tensors (Tetragonal-3)

Family	*Group*	*Material Tensors* $\overleftrightarrow{\varepsilon}$	$\overleftrightarrow{\mu}$	$\overleftrightarrow{\xi}$	$\overleftrightarrow{\zeta}$	[*# of parameters*]
Tetragonal	$\bar{4}'2m'$	$\begin{bmatrix}\varepsilon_{xx}&0&0\\0&\varepsilon_{xx}&0\\0&0&\varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx}&0&0\\0&\mu_{xx}&0\\0&0&\mu_{zz}\end{bmatrix}$	$\begin{bmatrix}\xi_{xx}&0&0\\0&\xi_{yy}&0\\0&0&\xi_{zz}\end{bmatrix}$	$\begin{bmatrix}\xi_{yy}&0&0\\0&\xi_{xx}&0\\0&0&\xi_{zz}\end{bmatrix}$	[7]
Tetragonal	$\bar{4}'m2'$	$\begin{bmatrix}\varepsilon_{xx}&0&0\\0&\varepsilon_{xx}&0\\0&0&\varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx}&0&0\\0&\mu_{xx}&0\\0&0&\mu_{zz}\end{bmatrix}$	$\begin{bmatrix}0&\xi_{xy}&0\\\xi_{yx}&0&0\\0&0&0\end{bmatrix}$	$\begin{bmatrix}0&-\xi_{xy}&0\\-\xi_{yx}&0&0\\0&0&0\end{bmatrix}$	[6]
Tetragonal	$\bar{4}2'm'$	$\begin{bmatrix}\varepsilon_{xx}&\varepsilon_{xy}&0\\-\varepsilon_{xy}&\varepsilon_{xx}&0\\0&0&\varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx}&\mu_{xy}&0\\-\mu_{xy}&\mu_{xx}&0\\0&0&\mu_{zz}\end{bmatrix}$	$\begin{bmatrix}\xi_{xx}&\xi_{xy}&0\\\xi_{xy}&-\xi_{xx}&0\\0&0&0\end{bmatrix}$	$\begin{bmatrix}-\xi_{xx}&\xi_{xy}&0\\\xi_{xy}&\xi_{xx}&0\\0&0&0\end{bmatrix}$	[8]
Tetragonal	$4/mmm1'$	$\begin{bmatrix}\varepsilon_{xx}&0&0\\0&\varepsilon_{xx}&0\\0&0&\varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx}&0&0\\0&\mu_{xx}&0\\0&0&\mu_{zz}\end{bmatrix}$	$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	[4]
Tetragonal	$4/mmm$	$\begin{bmatrix}\varepsilon_{xx}&0&0\\0&\varepsilon_{xx}&0\\0&0&\varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx}&0&0\\0&\mu_{xx}&0\\0&0&\mu_{zz}\end{bmatrix}$	$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	[4]
Tetragonal	$4'/mmm'$	$\begin{bmatrix}\varepsilon_{xx}&0&0\\0&\varepsilon_{xx}&0\\0&0&\varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx}&0&0\\0&\mu_{xx}&0\\0&0&\mu_{zz}\end{bmatrix}$	$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	[4]
Tetragonal	$4/mm'm'$	$\begin{bmatrix}\varepsilon_{xx}&\varepsilon_{xy}&0\\-\varepsilon_{xy}&\varepsilon_{xx}&0\\0&0&\varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx}&\mu_{xy}&0\\-\mu_{xy}&\mu_{xx}&0\\0&0&\mu_{zz}\end{bmatrix}$	$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	[6]
Tetragonal	$4/m'm'm'$	$\begin{bmatrix}\varepsilon_{xx}&0&0\\0&\varepsilon_{xx}&0\\0&0&\varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx}&0&0\\0&\mu_{xx}&0\\0&0&\mu_{zz}\end{bmatrix}$	$\begin{bmatrix}\xi_{xx}&0&0\\0&\xi_{xx}&0\\0&0&\xi_{zz}\end{bmatrix}$	$\begin{bmatrix}\xi_{xx}&0&0\\0&\xi_{xx}&0\\0&0&\xi_{zz}\end{bmatrix}$	[6]
Tetragonal	$4/m'mm$	$\begin{bmatrix}\varepsilon_{xx}&0&0\\0&\varepsilon_{xx}&0\\0&0&\varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx}&0&0\\0&\mu_{xx}&0\\0&0&\mu_{zz}\end{bmatrix}$	$\begin{bmatrix}0&\xi_{xy}&0\\-\xi_{xy}&0&0\\0&0&0\end{bmatrix}$	$\begin{bmatrix}0&-\xi_{xy}&0\\\xi_{xy}&0&0\\0&0&0\end{bmatrix}$	[5]
Tetragonal	$4'/m'm'm$	$\begin{bmatrix}\varepsilon_{xx}&0&0\\0&\varepsilon_{xx}&0\\0&0&\varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx}&0&0\\0&\mu_{xx}&0\\0&0&\mu_{zz}\end{bmatrix}$	$\begin{bmatrix}\xi_{xx}&0&0\\0&-\xi_{xx}&0\\0&0&0\end{bmatrix}$	$\begin{bmatrix}\xi_{xx}&0&0\\0&-\xi_{xx}&0\\0&0&0\end{bmatrix}$	[5]

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Appendix – Material Property Tensors (Trigonal-1)

Family	*Group*	*Material Tensors* $\overleftrightarrow{\varepsilon}$	$\overleftrightarrow{\mu}$	$\overleftrightarrow{\xi}$	$\overleftrightarrow{\zeta}$	[*# of parameters*]
Trigonal	$31'$	$\begin{bmatrix}\varepsilon_{xx}&0&0\\0&\varepsilon_{xx}&0\\0&0&\varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx}&0&0\\0&\mu_{xx}&0\\0&0&\mu_{zz}\end{bmatrix}$	$\begin{bmatrix}\xi_{xx}&\xi_{xy}&0\\-\xi_{xy}&\xi_{xx}&0\\0&0&\xi_{zz}\end{bmatrix}$	$\begin{bmatrix}-\xi_{xx}&\xi_{xy}&0\\-\xi_{xy}&-\xi_{xx}&0\\0&0&-\xi_{zz}\end{bmatrix}$	[7]
Trigonal	3	$\begin{bmatrix}\varepsilon_{xx}&\varepsilon_{xy}&0\\-\varepsilon_{xy}&\varepsilon_{xx}&0\\0&0&\varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx}&\mu_{xy}&0\\-\mu_{xy}&\mu_{xx}&0\\0&0&\mu_{zz}\end{bmatrix}$	$\begin{bmatrix}\xi_{xx}&\xi_{xy}&0\\-\xi_{xy}&\xi_{xx}&0\\0&0&\xi_{zz}\end{bmatrix}$	$\begin{bmatrix}\zeta_{xx}&\zeta_{xy}&0\\-\zeta_{xy}&\zeta_{xx}&0\\0&0&\zeta_{zz}\end{bmatrix}$	[12]
Trigonal	$\bar{3}1'$	$\begin{bmatrix}\varepsilon_{xx}&0&0\\0&\varepsilon_{xx}&0\\0&0&\varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx}&0&0\\0&\mu_{xx}&0\\0&0&\mu_{zz}\end{bmatrix}$	$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	[4]
Trigonal	$\bar{3}$	$\begin{bmatrix}\varepsilon_{xx}&\varepsilon_{xy}&0\\-\varepsilon_{xy}&\varepsilon_{xx}&0\\0&0&\varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx}&\mu_{xy}&0\\-\mu_{xy}&\mu_{xx}&0\\0&0&\mu_{zz}\end{bmatrix}$	$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	[6]
Trigonal	$\bar{3}'$	$\begin{bmatrix}\varepsilon_{xx}&0&0\\0&\varepsilon_{xx}&0\\0&0&\varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx}&0&0\\0&\mu_{xx}&0\\0&0&\mu_{zz}\end{bmatrix}$	$\begin{bmatrix}\xi_{xx}&\xi_{xy}&0\\-\xi_{xy}&\xi_{xx}&0\\0&0&\xi_{zz}\end{bmatrix}$	$\begin{bmatrix}\xi_{xx}&-\xi_{xy}&0\\\xi_{xy}&\xi_{xx}&0\\0&0&\xi_{zz}\end{bmatrix}$	[7]
Trigonal	$321'$	$\begin{bmatrix}\varepsilon_{xx}&0&0\\0&\varepsilon_{xx}&0\\0&0&\varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx}&0&0\\0&\mu_{xx}&0\\0&0&\mu_{zz}\end{bmatrix}$	$\begin{bmatrix}\xi_{xx}&0&0\\0&\xi_{xx}&0\\0&0&\xi_{zz}\end{bmatrix}$	$\begin{bmatrix}-\xi_{xx}&0&0\\0&-\xi_{xx}&0\\0&0&-\xi_{zz}\end{bmatrix}$	[6]
Trigonal	32	$\begin{bmatrix}\varepsilon_{xx}&0&0\\0&\varepsilon_{xx}&0\\0&0&\varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx}&0&0\\0&\mu_{xx}&0\\0&0&\mu_{zz}\end{bmatrix}$	$\begin{bmatrix}\xi_{xx}&0&0\\0&\xi_{xx}&0\\0&0&\xi_{zz}\end{bmatrix}$	$\begin{bmatrix}\zeta_{xx}&0&0\\0&\zeta_{xx}&0\\0&0&\zeta_{zz}\end{bmatrix}$	[8]
Trigonal	$32'$	$\begin{bmatrix}\varepsilon_{xx}&\varepsilon_{xy}&0\\-\varepsilon_{xy}&\varepsilon_{xx}&0\\0&0&\varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx}&\mu_{xy}&0\\-\mu_{xy}&\mu_{xx}&0\\0&0&\mu_{zz}\end{bmatrix}$	$\begin{bmatrix}\xi_{xx}&\xi_{xy}&0\\-\xi_{xy}&\xi_{xx}&0\\0&0&\xi_{zz}\end{bmatrix}$	$\begin{bmatrix}-\xi_{xx}&-\xi_{xy}&0\\\xi_{xy}&-\xi_{xx}&0\\0&0&-\xi_{zz}\end{bmatrix}$	[9]

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Appendix – Material Property Tensors (Trigonal-2)

Family	Group	$\overleftrightarrow{\varepsilon}$	$\overleftrightarrow{\mu}$	$\overleftrightarrow{\xi}$	$\overleftrightarrow{\zeta}$	[# of parameters]
		Material Tensors				
Trigonal	$3m1'$	$\begin{bmatrix}\varepsilon_{xx}&0&0\\0&\varepsilon_{xx}&0\\0&0&\varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx}&0&0\\0&\mu_{xx}&0\\0&0&\mu_{zz}\end{bmatrix}$	$\begin{bmatrix}0&\xi_{xy}&0\\-\xi_{xy}&0&0\\0&0&0\end{bmatrix}$	$\begin{bmatrix}0&\xi_{xy}&0\\-\xi_{xy}&0&0\\0&0&0\end{bmatrix}$	[5]
Trigonal	$3m$	$\begin{bmatrix}\varepsilon_{xx}&0&0\\0&\varepsilon_{xx}&0\\0&0&\varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx}&0&0\\0&\mu_{xx}&0\\0&0&\mu_{zz}\end{bmatrix}$	$\begin{bmatrix}0&\xi_{xy}&0\\-\xi_{xy}&0&0\\0&0&0\end{bmatrix}$	$\begin{bmatrix}0&\zeta_{xy}&0\\-\zeta_{xy}&0&0\\0&0&0\end{bmatrix}$	[6]
Trigonal	$3m'$	$\begin{bmatrix}\varepsilon_{xx}&\varepsilon_{xy}&0\\-\varepsilon_{xy}&\varepsilon_{xx}&0\\0&0&\varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx}&\mu_{xy}&0\\-\mu_{xy}&\mu_{xx}&0\\0&0&\mu_{zz}\end{bmatrix}$	$\begin{bmatrix}\xi_{xx}&\xi_{xy}&0\\-\xi_{xy}&\xi_{xx}&0\\0&0&\xi_{zz}\end{bmatrix}$	$\begin{bmatrix}\xi_{xx}&\xi_{xy}&0\\-\xi_{xy}&\xi_{xx}&0\\0&0&\xi_{zz}\end{bmatrix}$	[9]
Trigonal	$\bar{3}m1'$	$\begin{bmatrix}\varepsilon_{xx}&0&0\\0&\varepsilon_{xx}&0\\0&0&\varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx}&0&0\\0&\mu_{xx}&0\\0&0&\mu_{zz}\end{bmatrix}$	$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	[4]
Trigonal	$\bar{3}m$	$\begin{bmatrix}\varepsilon_{xx}&0&0\\0&\varepsilon_{xx}&0\\0&0&\varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx}&0&0\\0&\mu_{xx}&0\\0&0&\mu_{zz}\end{bmatrix}$	$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	[4]
Trigonal	$\bar{3}m'$	$\begin{bmatrix}\varepsilon_{xx}&\varepsilon_{xy}&0\\-\varepsilon_{xy}&\varepsilon_{xx}&0\\0&0&\varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx}&\mu_{xy}&0\\-\mu_{xy}&\mu_{xx}&0\\0&0&\mu_{zz}\end{bmatrix}$	$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	[6]
Trigonal	$\bar{3}'m'$	$\begin{bmatrix}\varepsilon_{xx}&0&0\\0&\varepsilon_{xx}&0\\0&0&\varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx}&0&0\\0&\mu_{xx}&0\\0&0&\mu_{zz}\end{bmatrix}$	$\begin{bmatrix}\xi_{xx}&0&0\\0&\xi_{xx}&0\\0&0&\xi_{zz}\end{bmatrix}$	$\begin{bmatrix}\xi_{xx}&0&0\\0&\xi_{xx}&0\\0&0&\xi_{zz}\end{bmatrix}$	[6]
Trigonal	$\bar{3}'m$	$\begin{bmatrix}\varepsilon_{xx}&0&0\\0&\varepsilon_{xx}&0\\0&0&\varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx}&0&0\\0&\mu_{xx}&0\\0&0&\mu_{zz}\end{bmatrix}$	$\begin{bmatrix}0&\xi_{xy}&0\\-\xi_{xy}&0&0\\0&0&0\end{bmatrix}$	$\begin{bmatrix}0&-\xi_{xy}&0\\\xi_{xy}&0&0\\0&0&0\end{bmatrix}$	[5]

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Appendix – Material Property Tensors (Hexagonal-1)

Family	Group	$\overleftrightarrow{\varepsilon}$	$\overleftrightarrow{\mu}$	$\overleftrightarrow{\xi}$	$\overleftrightarrow{\zeta}$	[# of parameters]
		Material Tensors				
Hexagonal	$61'$	$\begin{bmatrix}\varepsilon_{xx}&0&0\\0&\varepsilon_{xx}&0\\0&0&\varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx}&0&0\\0&\mu_{xx}&0\\0&0&\mu_{zz}\end{bmatrix}$	$\begin{bmatrix}\xi_{xx}&\xi_{xy}&0\\-\xi_{xy}&\xi_{xx}&0\\0&0&\xi_{zz}\end{bmatrix}$	$\begin{bmatrix}-\xi_{xx}&\xi_{xy}&0\\-\xi_{xy}&-\xi_{xx}&0\\0&0&-\xi_{zz}\end{bmatrix}$	[7]
Hexagonal	6	$\begin{bmatrix}\varepsilon_{xx}&\varepsilon_{xy}&0\\-\varepsilon_{xy}&\varepsilon_{xx}&0\\0&0&\varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx}&\mu_{xy}&0\\-\mu_{xy}&\mu_{xx}&0\\0&0&\mu_{zz}\end{bmatrix}$	$\begin{bmatrix}\xi_{xx}&\xi_{xy}&0\\-\xi_{xy}&\xi_{xx}&0\\0&0&\xi_{zz}\end{bmatrix}$	$\begin{bmatrix}\zeta_{xx}&\zeta_{xy}&0\\-\zeta_{xy}&\zeta_{xx}&0\\0&0&\zeta_{zz}\end{bmatrix}$	[12]
Hexagonal	$6'$	$\begin{bmatrix}\varepsilon_{xx}&0&0\\0&\varepsilon_{xx}&0\\0&0&\varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx}&0&0\\0&\mu_{xx}&0\\0&0&\mu_{zz}\end{bmatrix}$	$\begin{bmatrix}\xi_{xx}&\xi_{xy}&0\\-\xi_{xy}&\xi_{xx}&0\\0&0&\xi_{zz}\end{bmatrix}$	$\begin{bmatrix}-\xi_{xx}&\xi_{xy}&0\\-\xi_{xy}&-\xi_{xx}&0\\0&0&-\xi_{zz}\end{bmatrix}$	[7]
Hexagonal	$\bar{6}1'$	$\begin{bmatrix}\varepsilon_{xx}&0&0\\0&\varepsilon_{xx}&0\\0&0&\varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx}&0&0\\0&\mu_{xx}&0\\0&0&\mu_{zz}\end{bmatrix}$	$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	[4]
Hexagonal	$\bar{6}$	$\begin{bmatrix}\varepsilon_{xx}&\varepsilon_{xy}&0\\-\varepsilon_{xy}&\varepsilon_{xx}&0\\0&0&\varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx}&\mu_{xy}&0\\-\mu_{xy}&\mu_{xx}&0\\0&0&\mu_{zz}\end{bmatrix}$	$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	[6]
Hexagonal	$\bar{6}'$	$\begin{bmatrix}\varepsilon_{xx}&0&0\\0&\varepsilon_{xx}&0\\0&0&\varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx}&0&0\\0&\mu_{xx}&0\\0&0&\mu_{zz}\end{bmatrix}$	$\begin{bmatrix}\xi_{xx}&\xi_{xy}&0\\-\xi_{xy}&\xi_{xx}&0\\0&0&\xi_{zz}\end{bmatrix}$	$\begin{bmatrix}\xi_{xx}&-\xi_{xy}&0\\\xi_{xy}&\xi_{xx}&0\\0&0&\xi_{zz}\end{bmatrix}$	[7]
Hexagonal	$6/m1'$	$\begin{bmatrix}\varepsilon_{xx}&0&0\\0&\varepsilon_{xx}&0\\0&0&\varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx}&0&0\\0&\mu_{xx}&0\\0&0&\mu_{zz}\end{bmatrix}$	$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	[4]
Hexagonal	$6/m$	$\begin{bmatrix}\varepsilon_{xx}&\varepsilon_{xy}&0\\-\varepsilon_{xy}&\varepsilon_{xx}&0\\0&0&\varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx}&\mu_{xy}&0\\-\mu_{xy}&\mu_{xx}&0\\0&0&\mu_{zz}\end{bmatrix}$	$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	[6]
Hexagonal	$6'/m'$	$\begin{bmatrix}\varepsilon_{xx}&0&0\\0&\varepsilon_{xx}&0\\0&0&\varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx}&0&0\\0&\mu_{xx}&0\\0&0&\mu_{zz}\end{bmatrix}$	$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	[4]
Hexagonal	$6/m'$	$\begin{bmatrix}\varepsilon_{xx}&0&0\\0&\varepsilon_{xx}&0\\0&0&\varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx}&0&0\\0&\mu_{xx}&0\\0&0&\mu_{zz}\end{bmatrix}$	$\begin{bmatrix}\xi_{xx}&\xi_{xy}&0\\-\xi_{xy}&\xi_{xx}&0\\0&0&\xi_{zz}\end{bmatrix}$	$\begin{bmatrix}\xi_{xx}&-\xi_{xy}&0\\\xi_{xy}&\xi_{xx}&0\\0&0&\xi_{zz}\end{bmatrix}$	[7]
Hexagonal	$6'/m$	$\begin{bmatrix}\varepsilon_{xx}&0&0\\0&\varepsilon_{xx}&0\\0&0&\varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx}&0&0\\0&\mu_{xx}&0\\0&0&\mu_{zz}\end{bmatrix}$	$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	[4]

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Appendix – Material Property Tensors (Hexagonal-2)

Family	*Group*	Material Tensors: $\overleftrightarrow{\varepsilon}$	$\overleftrightarrow{\mu}$	$\overleftrightarrow{\xi}$	$\overleftrightarrow{\zeta}$	[# of parameters]
Hexagonal	$6221'$	$\begin{bmatrix}\varepsilon_{xx}&0&0\\0&\varepsilon_{xx}&0\\0&0&\varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx}&0&0\\0&\mu_{xx}&0\\0&0&\mu_{zz}\end{bmatrix}$	$\begin{bmatrix}\xi_{xx}&0&0\\0&\xi_{xx}&0\\0&0&\xi_{zz}\end{bmatrix}$	$\begin{bmatrix}-\xi_{xx}&0&0\\0&-\xi_{xx}&0\\0&0&-\xi_{zz}\end{bmatrix}$	[6]
Hexagonal	622	$\begin{bmatrix}\varepsilon_{xx}&0&0\\0&\varepsilon_{xx}&0\\0&0&\varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx}&0&0\\0&\mu_{xx}&0\\0&0&\mu_{zz}\end{bmatrix}$	$\begin{bmatrix}\xi_{xx}&0&0\\0&\xi_{xx}&0\\0&0&\xi_{zz}\end{bmatrix}$	$\begin{bmatrix}\zeta_{xx}&0&0\\0&\zeta_{xx}&0\\0&0&\zeta_{zz}\end{bmatrix}$	[8]
Hexagonal	$6'22'$	$\begin{bmatrix}\varepsilon_{xx}&0&0\\0&\varepsilon_{xx}&0\\0&0&\varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx}&0&0\\0&\mu_{xx}&0\\0&0&\mu_{zz}\end{bmatrix}$	$\begin{bmatrix}\xi_{xx}&0&0\\0&\xi_{xx}&0\\0&0&\xi_{zz}\end{bmatrix}$	$\begin{bmatrix}-\xi_{xx}&0&0\\0&-\xi_{xx}&0\\0&0&-\xi_{zz}\end{bmatrix}$	[6]
Hexagonal	$62'2'$	$\begin{bmatrix}\varepsilon_{xx}&\varepsilon_{xy}&0\\-\varepsilon_{xy}&\varepsilon_{xx}&0\\0&0&\varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx}&\mu_{xy}&0\\-\mu_{xy}&\mu_{xx}&0\\0&0&\mu_{zz}\end{bmatrix}$	$\begin{bmatrix}\xi_{xx}&\xi_{xy}&0\\-\xi_{xy}&\xi_{xx}&0\\0&0&\xi_{zz}\end{bmatrix}$	$\begin{bmatrix}-\xi_{xx}&-\xi_{xy}&0\\\xi_{xy}&-\xi_{xx}&0\\0&0&-\xi_{zz}\end{bmatrix}$	[9]
Hexagonal	$6mm1'$	$\begin{bmatrix}\varepsilon_{xx}&0&0\\0&\varepsilon_{xx}&0\\0&0&\varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx}&0&0\\0&\mu_{xx}&0\\0&0&\mu_{zz}\end{bmatrix}$	$\begin{bmatrix}0&\xi_{xy}&0\\-\xi_{xy}&0&0\\0&0&0\end{bmatrix}$	$\begin{bmatrix}0&\xi_{xy}&0\\-\xi_{xy}&0&0\\0&0&0\end{bmatrix}$	[5]
Hexagonal	$6mm$	$\begin{bmatrix}\varepsilon_{xx}&0&0\\0&\varepsilon_{xx}&0\\0&0&\varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx}&0&0\\0&\mu_{xx}&0\\0&0&\mu_{zz}\end{bmatrix}$	$\begin{bmatrix}0&\xi_{xy}&0\\-\xi_{xy}&0&0\\0&0&0\end{bmatrix}$	$\begin{bmatrix}0&\zeta_{xy}&0\\-\zeta_{xy}&0&0\\0&0&0\end{bmatrix}$	[6]
Hexagonal	$6'mm'$	$\begin{bmatrix}\varepsilon_{xx}&0&0\\0&\varepsilon_{xx}&0\\0&0&\varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx}&0&0\\0&\mu_{xx}&0\\0&0&\mu_{zz}\end{bmatrix}$	$\begin{bmatrix}0&\xi_{xy}&0\\-\xi_{xy}&0&0\\0&0&0\end{bmatrix}$	$\begin{bmatrix}0&\xi_{xy}&0\\-\xi_{xy}&0&0\\0&0&0\end{bmatrix}$	[5]
Hexagonal	$6m'm'$	$\begin{bmatrix}\varepsilon_{xx}&\varepsilon_{xy}&0\\-\varepsilon_{xy}&\varepsilon_{xx}&0\\0&0&\varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx}&\mu_{xy}&0\\-\mu_{xy}&\mu_{xx}&0\\0&0&\mu_{zz}\end{bmatrix}$	$\begin{bmatrix}\xi_{xx}&\xi_{xy}&0\\-\xi_{xy}&\xi_{xx}&0\\0&0&\xi_{zz}\end{bmatrix}$	$\begin{bmatrix}\xi_{xx}&\xi_{xy}&0\\-\xi_{xy}&\xi_{xx}&0\\0&0&\xi_{zz}\end{bmatrix}$	[9]
Hexagonal	$\bar{6}m21'$	$\begin{bmatrix}\varepsilon_{xx}&0&0\\0&\varepsilon_{xx}&0\\0&0&\varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx}&0&0\\0&\mu_{xx}&0\\0&0&\mu_{zz}\end{bmatrix}$	$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	[4]
Hexagonal	$\bar{6}m2$	$\begin{bmatrix}\varepsilon_{xx}&0&0\\0&\varepsilon_{xx}&0\\0&0&\varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx}&0&0\\0&\mu_{xx}&0\\0&0&\mu_{zz}\end{bmatrix}$	$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	[4]

Appendix – Material Property Tensors (Hexagonal-3)

Family	*Group*	Material Tensors: $\overleftrightarrow{\varepsilon}$	$\overleftrightarrow{\mu}$	$\overleftrightarrow{\xi}$	$\overleftrightarrow{\zeta}$	[# of parameters]
Hexagonal	$\bar{6}'2m'$	$\begin{bmatrix}\varepsilon_{xx}&0&0\\0&\varepsilon_{xx}&0\\0&0&\varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx}&0&0\\0&\mu_{xx}&0\\0&0&\mu_{zz}\end{bmatrix}$	$\begin{bmatrix}\xi_{xx}&0&0\\0&\xi_{xx}&0\\0&0&\xi_{zz}\end{bmatrix}$	$\begin{bmatrix}\xi_{xx}&0&0\\0&\xi_{xx}&0\\0&0&\xi_{zz}\end{bmatrix}$	[6]
Hexagonal	$\bar{6}'m2'$	$\begin{bmatrix}\varepsilon_{xx}&0&0\\0&\varepsilon_{xx}&0\\0&0&\varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx}&0&0\\0&\mu_{xx}&0\\0&0&\mu_{zz}\end{bmatrix}$	$\begin{bmatrix}0&\xi_{xy}&0\\-\xi_{xy}&0&0\\0&0&0\end{bmatrix}$	$\begin{bmatrix}0&-\xi_{xy}&0\\\xi_{xy}&0&0\\0&0&0\end{bmatrix}$	[5]
Hexagonal	$\bar{6}m'2'$	$\begin{bmatrix}\varepsilon_{xx}&\varepsilon_{xy}&0\\-\varepsilon_{xy}&\varepsilon_{xx}&0\\0&0&\varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx}&\mu_{xy}&0\\-\mu_{xy}&\mu_{xx}&0\\0&0&\mu_{zz}\end{bmatrix}$	$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	[6]
Hexagonal	$6/mmm1'$	$\begin{bmatrix}\varepsilon_{xx}&0&0\\0&\varepsilon_{xx}&0\\0&0&\varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx}&0&0\\0&\mu_{xx}&0\\0&0&\mu_{zz}\end{bmatrix}$	$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	[4]
Hexagonal	$6/mmm$	$\begin{bmatrix}\varepsilon_{xx}&0&0\\0&\varepsilon_{xx}&0\\0&0&\varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx}&0&0\\0&\mu_{xx}&0\\0&0&\mu_{zz}\end{bmatrix}$	$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	[4]
Hexagonal	$6'/m'mm'$	$\begin{bmatrix}\varepsilon_{xx}&0&0\\0&\varepsilon_{xx}&0\\0&0&\varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx}&0&0\\0&\mu_{xx}&0\\0&0&\mu_{zz}\end{bmatrix}$	$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	[4]
Hexagonal	$6/mm'm'$	$\begin{bmatrix}\varepsilon_{xx}&\varepsilon_{xy}&0\\-\varepsilon_{xy}&\varepsilon_{xx}&0\\0&0&\varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx}&\mu_{xy}&0\\-\mu_{xy}&\mu_{xx}&0\\0&0&\mu_{zz}\end{bmatrix}$	$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	[6]
Hexagonal	$6/m'm'm'$	$\begin{bmatrix}\varepsilon_{xx}&0&0\\0&\varepsilon_{xx}&0\\0&0&\varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx}&0&0\\0&\mu_{xx}&0\\0&0&\mu_{zz}\end{bmatrix}$	$\begin{bmatrix}\xi_{xx}&0&0\\0&\xi_{xx}&0\\0&0&\xi_{zz}\end{bmatrix}$	$\begin{bmatrix}\xi_{xx}&0&0\\0&\xi_{xx}&0\\0&0&\xi_{zz}\end{bmatrix}$	[6]
Hexagonal	$6/m'mm$	$\begin{bmatrix}\varepsilon_{xx}&0&0\\0&\varepsilon_{xx}&0\\0&0&\varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx}&0&0\\0&\mu_{xx}&0\\0&0&\mu_{zz}\end{bmatrix}$	$\begin{bmatrix}0&\xi_{xy}&0\\-\xi_{xy}&0&0\\0&0&0\end{bmatrix}$	$\begin{bmatrix}0&-\xi_{xy}&0\\\xi_{xy}&0&0\\0&0&0\end{bmatrix}$	[5]
Hexagonal	$6'/mmm'$	$\begin{bmatrix}\varepsilon_{xx}&0&0\\0&\varepsilon_{xx}&0\\0&0&\varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx}&0&0\\0&\mu_{xx}&0\\0&0&\mu_{zz}\end{bmatrix}$	$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	[4]

Appendix – Material Property Tensors (Cubic-1)

Family	Group	Material Tensors $\bar{\bar{\varepsilon}}$	$\bar{\bar{\mu}}$	$\bar{\bar{\xi}}$	$\bar{\bar{\zeta}}$	[# of parameters]
Cubic	$231'$	$\begin{bmatrix}\varepsilon&0&0\\0&\varepsilon&0\\0&0&\varepsilon\end{bmatrix}$	$\begin{bmatrix}\mu&0&0\\0&\mu&0\\0&0&\mu\end{bmatrix}$	$\begin{bmatrix}\xi&0&0\\0&\xi&0\\0&0&\xi\end{bmatrix}$	$\begin{bmatrix}-\xi&0&0\\0&-\xi&0\\0&0&-\xi\end{bmatrix}$	[3]
Cubic	23	$\begin{bmatrix}\varepsilon&0&0\\0&\varepsilon&0\\0&0&\varepsilon\end{bmatrix}$	$\begin{bmatrix}\mu&0&0\\0&\mu&0\\0&0&\mu\end{bmatrix}$	$\begin{bmatrix}\xi&0&0\\0&\xi&0\\0&0&\xi\end{bmatrix}$	$\begin{bmatrix}\zeta&0&0\\0&\zeta&0\\0&0&\zeta\end{bmatrix}$	[4]
Cubic	$m\bar{3}1'$	$\begin{bmatrix}\varepsilon&0&0\\0&\varepsilon&0\\0&0&\varepsilon\end{bmatrix}$	$\begin{bmatrix}\mu&0&0\\0&\mu&0\\0&0&\mu\end{bmatrix}$	$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	[2]
Cubic	$m\bar{3}$	$\begin{bmatrix}\varepsilon&0&0\\0&\varepsilon&0\\0&0&\varepsilon\end{bmatrix}$	$\begin{bmatrix}\mu&0&0\\0&\mu&0\\0&0&\mu\end{bmatrix}$	$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	[2]
Cubic	$m'\bar{3}'$	$\begin{bmatrix}\varepsilon&0&0\\0&\varepsilon&0\\0&0&\varepsilon\end{bmatrix}$	$\begin{bmatrix}\mu&0&0\\0&\mu&0\\0&0&\mu\end{bmatrix}$	$\begin{bmatrix}\xi&0&0\\0&\xi&0\\0&0&\xi\end{bmatrix}$	$\begin{bmatrix}\xi&0&0\\0&\xi&0\\0&0&\xi\end{bmatrix}$	[3]
Cubic	$4321'$	$\begin{bmatrix}\varepsilon&0&0\\0&\varepsilon&0\\0&0&\varepsilon\end{bmatrix}$	$\begin{bmatrix}\mu&0&0\\0&\mu&0\\0&0&\mu\end{bmatrix}$	$\begin{bmatrix}\xi&0&0\\0&\xi&0\\0&0&\xi\end{bmatrix}$	$\begin{bmatrix}-\xi&0&0\\0&-\xi&0\\0&0&-\xi\end{bmatrix}$	[3]
Cubic	432	$\begin{bmatrix}\varepsilon&0&0\\0&\varepsilon&0\\0&0&\varepsilon\end{bmatrix}$	$\begin{bmatrix}\mu&0&0\\0&\mu&0\\0&0&\mu\end{bmatrix}$	$\begin{bmatrix}\xi&0&0\\0&\xi&0\\0&0&\xi\end{bmatrix}$	$\begin{bmatrix}\zeta&0&0\\0&\zeta&0\\0&0&\zeta\end{bmatrix}$	[4]
Cubic	$4'32'$	$\begin{bmatrix}\varepsilon&0&0\\0&\varepsilon&0\\0&0&\varepsilon\end{bmatrix}$	$\begin{bmatrix}\mu&0&0\\0&\mu&0\\0&0&\mu\end{bmatrix}$	$\begin{bmatrix}\xi&0&0\\0&\xi&0\\0&0&\xi\end{bmatrix}$	$\begin{bmatrix}-\xi&0&0\\0&-\xi&0\\0&0&-\xi\end{bmatrix}$	[3]

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Appendix – Material Property Tensors (Cubic-2)

Family	Group	Material Tensors $\bar{\bar{\varepsilon}}$	$\bar{\bar{\mu}}$	$\bar{\bar{\xi}}$	$\bar{\bar{\zeta}}$	[# of parameters]
Cubic	$\bar{4}3m1'$	$\begin{bmatrix}\varepsilon&0&0\\0&\varepsilon&0\\0&0&\varepsilon\end{bmatrix}$	$\begin{bmatrix}\mu&0&0\\0&\mu&0\\0&0&\mu\end{bmatrix}$	$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	[2]
Cubic	$\bar{4}3m$	$\begin{bmatrix}\varepsilon&0&0\\0&\varepsilon&0\\0&0&\varepsilon\end{bmatrix}$	$\begin{bmatrix}\mu&0&0\\0&\mu&0\\0&0&\mu\end{bmatrix}$	$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	[2]
Cubic	$\bar{4}'3m'$	$\begin{bmatrix}\varepsilon&0&0\\0&\varepsilon&0\\0&0&\varepsilon\end{bmatrix}$	$\begin{bmatrix}\mu&0&0\\0&\mu&0\\0&0&\mu\end{bmatrix}$	$\begin{bmatrix}\xi&0&0\\0&\xi&0\\0&0&\xi\end{bmatrix}$	$\begin{bmatrix}\xi&0&0\\0&\xi&0\\0&0&\xi\end{bmatrix}$	[3]
Cubic	$m\bar{3}m1'$	$\begin{bmatrix}\varepsilon&0&0\\0&\varepsilon&0\\0&0&\varepsilon\end{bmatrix}$	$\begin{bmatrix}\mu&0&0\\0&\mu&0\\0&0&\mu\end{bmatrix}$	$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	[2]
Cubic	$m\bar{3}m$	$\begin{bmatrix}\varepsilon&0&0\\0&\varepsilon&0\\0&0&\varepsilon\end{bmatrix}$	$\begin{bmatrix}\mu&0&0\\0&\mu&0\\0&0&\mu\end{bmatrix}$	$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	[2]
Cubic	$m\bar{3}m'$	$\begin{bmatrix}\varepsilon&0&0\\0&\varepsilon&0\\0&0&\varepsilon\end{bmatrix}$	$\begin{bmatrix}\mu&0&0\\0&\mu&0\\0&0&\mu\end{bmatrix}$	$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	[2]
Cubic	$m'\bar{3}'m'$	$\begin{bmatrix}\varepsilon&0&0\\0&\varepsilon&0\\0&0&\varepsilon\end{bmatrix}$	$\begin{bmatrix}\mu&0&0\\0&\mu&0\\0&0&\mu\end{bmatrix}$	$\begin{bmatrix}\xi&0&0\\0&\xi&0\\0&0&\xi\end{bmatrix}$	$\begin{bmatrix}\xi&0&0\\0&\xi&0\\0&0&\xi\end{bmatrix}$	[3]
Cubic	$m'\bar{3}'m$	$\begin{bmatrix}\varepsilon&0&0\\0&\varepsilon&0\\0&0&\varepsilon\end{bmatrix}$	$\begin{bmatrix}\mu&0&0\\0&\mu&0\\0&0&\mu\end{bmatrix}$	$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	[2]

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Appendix – Material Property Tensors (Continuous-1)

Family	*Group*	*Material Tensors* $\overleftrightarrow{\varepsilon}$	$\overleftrightarrow{\mu}$	$\overleftrightarrow{\xi}$	$\overleftrightarrow{\zeta}$	[*# of parameters*]
Continuous	$\infty 1'$	$\begin{bmatrix}\varepsilon_{xx} & 0 & 0\\ 0 & \varepsilon_{xx} & 0\\ 0 & 0 & \varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx} & 0 & 0\\ 0 & \mu_{xx} & 0\\ 0 & 0 & \mu_{zz}\end{bmatrix}$	$\begin{bmatrix}\xi_{xx} & \xi_{xy} & 0\\ -\xi_{xy} & \xi_{xx} & 0\\ 0 & 0 & \xi_{zz}\end{bmatrix}$	$\begin{bmatrix}-\xi_{xx} & \xi_{xy} & 0\\ -\xi_{xy} & -\xi_{xx} & 0\\ 0 & 0 & -\xi_{zz}\end{bmatrix}$	[7]
Continuous	∞	$\begin{bmatrix}\varepsilon_{xx} & \varepsilon_{xy} & 0\\ -\varepsilon_{xy} & \varepsilon_{xx} & 0\\ 0 & 0 & \varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx} & \mu_{xy} & 0\\ -\mu_{xy} & \mu_{xx} & 0\\ 0 & 0 & \mu_{zz}\end{bmatrix}$	$\begin{bmatrix}\xi_{xx} & \xi_{xy} & 0\\ -\xi_{xy} & \xi_{xx} & 0\\ 0 & 0 & \xi_{zz}\end{bmatrix}$	$\begin{bmatrix}\zeta_{xx} & \zeta_{xy} & 0\\ -\zeta_{xy} & \zeta_{xx} & 0\\ 0 & 0 & \zeta_{zz}\end{bmatrix}$	[12]
Continuous	$\infty/m1'$	$\begin{bmatrix}\varepsilon_{xx} & 0 & 0\\ 0 & \varepsilon_{xx} & 0\\ 0 & 0 & \varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx} & 0 & 0\\ 0 & \mu_{xx} & 0\\ 0 & 0 & \mu_{zz}\end{bmatrix}$	$\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$	$\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$	[4]
Continuous	∞/m	$\begin{bmatrix}\varepsilon_{xx} & \varepsilon_{xy} & 0\\ -\varepsilon_{xy} & \varepsilon_{xx} & 0\\ 0 & 0 & \varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx} & \mu_{xy} & 0\\ -\mu_{xy} & \mu_{xx} & 0\\ 0 & 0 & \mu_{zz}\end{bmatrix}$	$\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$	$\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$	[6]
Continuous	∞/m'	$\begin{bmatrix}\varepsilon_{xx} & 0 & 0\\ 0 & \varepsilon_{xx} & 0\\ 0 & 0 & \varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx} & 0 & 0\\ 0 & \mu_{xx} & 0\\ 0 & 0 & \mu_{zz}\end{bmatrix}$	$\begin{bmatrix}\xi_{xx} & \xi_{xy} & 0\\ -\xi_{xy} & \xi_{xx} & 0\\ 0 & 0 & \xi_{zz}\end{bmatrix}$	$\begin{bmatrix}\xi_{xx} & -\xi_{xy} & 0\\ \xi_{xy} & \xi_{xx} & 0\\ 0 & 0 & \xi_{zz}\end{bmatrix}$	[7]
Continuous	$\infty 21'$	$\begin{bmatrix}\varepsilon_{xx} & 0 & 0\\ 0 & \varepsilon_{xx} & 0\\ 0 & 0 & \varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx} & 0 & 0\\ 0 & \mu_{xx} & 0\\ 0 & 0 & \mu_{zz}\end{bmatrix}$	$\begin{bmatrix}\xi_{xx} & 0 & 0\\ 0 & \xi_{xx} & 0\\ 0 & 0 & \xi_{zz}\end{bmatrix}$	$\begin{bmatrix}-\xi_{xx} & 0 & 0\\ 0 & -\xi_{xx} & 0\\ 0 & 0 & -\xi_{zz}\end{bmatrix}$	[6]
Continuous	$\infty 2$	$\begin{bmatrix}\varepsilon_{xx} & 0 & 0\\ 0 & \varepsilon_{xx} & 0\\ 0 & 0 & \varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx} & 0 & 0\\ 0 & \mu_{xx} & 0\\ 0 & 0 & \mu_{zz}\end{bmatrix}$	$\begin{bmatrix}\xi_{xx} & 0 & 0\\ 0 & \xi_{xx} & 0\\ 0 & 0 & \xi_{zz}\end{bmatrix}$	$\begin{bmatrix}\zeta_{xx} & 0 & 0\\ 0 & \zeta_{xx} & 0\\ 0 & 0 & \zeta_{zz}\end{bmatrix}$	[8]
Continuous	$\infty 2'$	$\begin{bmatrix}\varepsilon_{xx} & \varepsilon_{xy} & 0\\ -\varepsilon_{xy} & \varepsilon_{xx} & 0\\ 0 & 0 & \varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx} & \mu_{xy} & 0\\ -\mu_{xy} & \mu_{xx} & 0\\ 0 & 0 & \mu_{zz}\end{bmatrix}$	$\begin{bmatrix}\xi_{xx} & \xi_{xy} & 0\\ -\xi_{xy} & \xi_{xx} & 0\\ 0 & 0 & \xi_{zz}\end{bmatrix}$	$\begin{bmatrix}-\xi_{xx} & -\xi_{xy} & 0\\ \xi_{xy} & -\xi_{xx} & 0\\ 0 & 0 & -\xi_{zz}\end{bmatrix}$	[9]
Continuous	$\infty m1'$	$\begin{bmatrix}\varepsilon_{xx} & 0 & 0\\ 0 & \varepsilon_{xx} & 0\\ 0 & 0 & \varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx} & 0 & 0\\ 0 & \mu_{xx} & 0\\ 0 & 0 & \mu_{zz}\end{bmatrix}$	$\begin{bmatrix}0 & \xi_{xy} & 0\\ -\xi_{xy} & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$	$\begin{bmatrix}0 & \xi_{xy} & 0\\ -\xi_{xy} & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$	[5]
Continuous	∞m	$\begin{bmatrix}\varepsilon_{xx} & 0 & 0\\ 0 & \varepsilon_{xx} & 0\\ 0 & 0 & \varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx} & 0 & 0\\ 0 & \mu_{xx} & 0\\ 0 & 0 & \mu_{zz}\end{bmatrix}$	$\begin{bmatrix}0 & \xi_{xy} & 0\\ -\xi_{xy} & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$	$\begin{bmatrix}0 & \zeta_{xy} & 0\\ -\zeta_{xy} & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$	[6]
Continuous	$\infty m'$	$\begin{bmatrix}\varepsilon_{xx} & \varepsilon_{xy} & 0\\ -\varepsilon_{xy} & \varepsilon_{xx} & 0\\ 0 & 0 & \varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx} & \mu_{xy} & 0\\ -\mu_{xy} & \mu_{xx} & 0\\ 0 & 0 & \mu_{zz}\end{bmatrix}$	$\begin{bmatrix}\xi_{xx} & \xi_{xy} & 0\\ -\xi_{xy} & \xi_{xx} & 0\\ 0 & 0 & \xi_{zz}\end{bmatrix}$	$\begin{bmatrix}\xi_{xx} & \xi_{xy} & 0\\ -\xi_{xy} & \xi_{xx} & 0\\ 0 & 0 & \xi_{zz}\end{bmatrix}$	[9]

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Appendix – Material Property Tensors (Continuous-2)

Family	*Group*	*Material Tensors* $\overleftrightarrow{\varepsilon}$	$\overleftrightarrow{\mu}$	$\overleftrightarrow{\xi}$	$\overleftrightarrow{\zeta}$	[*# of parameters*]
Continuous	$\infty/mm1'$	$\begin{bmatrix}\varepsilon_{xx} & 0 & 0\\ 0 & \varepsilon_{xx} & 0\\ 0 & 0 & \varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx} & 0 & 0\\ 0 & \mu_{xx} & 0\\ 0 & 0 & \mu_{zz}\end{bmatrix}$	$\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$	$\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$	[4]
Continuous	∞/mm	$\begin{bmatrix}\varepsilon_{xx} & 0 & 0\\ 0 & \varepsilon_{xx} & 0\\ 0 & 0 & \varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx} & 0 & 0\\ 0 & \mu_{xx} & 0\\ 0 & 0 & \mu_{zz}\end{bmatrix}$	$\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$	$\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$	[4]
Continuous	∞/mm'	$\begin{bmatrix}\varepsilon_{xx} & \varepsilon_{xy} & 0\\ -\varepsilon_{xy} & \varepsilon_{xx} & 0\\ 0 & 0 & \varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx} & \mu_{xy} & 0\\ -\mu_{xy} & \mu_{xx} & 0\\ 0 & 0 & \mu_{zz}\end{bmatrix}$	$\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$	$\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$	[6]
Continuous	$\infty/m'm$	$\begin{bmatrix}\varepsilon_{xx} & 0 & 0\\ 0 & \varepsilon_{xx} & 0\\ 0 & 0 & \varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx} & 0 & 0\\ 0 & \mu_{xx} & 0\\ 0 & 0 & \mu_{zz}\end{bmatrix}$	$\begin{bmatrix}0 & \xi_{xy} & 0\\ -\xi_{xy} & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$	$\begin{bmatrix}0 & -\xi_{xy} & 0\\ \xi_{xy} & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$	[5]
Continuous	$\infty/m'm'$	$\begin{bmatrix}\varepsilon_{xx} & 0 & 0\\ 0 & \varepsilon_{xx} & 0\\ 0 & 0 & \varepsilon_{zz}\end{bmatrix}$	$\begin{bmatrix}\mu_{xx} & 0 & 0\\ 0 & \mu_{xx} & 0\\ 0 & 0 & \mu_{zz}\end{bmatrix}$	$\begin{bmatrix}\xi_{xx} & 0 & 0\\ 0 & \xi_{xx} & 0\\ 0 & 0 & \xi_{zz}\end{bmatrix}$	$\begin{bmatrix}\xi_{xx} & 0 & 0\\ 0 & \xi_{xx} & 0\\ 0 & 0 & \xi_{zz}\end{bmatrix}$	[6]
Continuous	$\infty\infty 1'$	$\begin{bmatrix}\varepsilon & 0 & 0\\ 0 & \varepsilon & 0\\ 0 & 0 & \varepsilon\end{bmatrix}$	$\begin{bmatrix}\mu & 0 & 0\\ 0 & \mu & 0\\ 0 & 0 & \mu\end{bmatrix}$	$\begin{bmatrix}\xi & 0 & 0\\ 0 & \xi & 0\\ 0 & 0 & \xi\end{bmatrix}$	$\begin{bmatrix}-\xi & 0 & 0\\ 0 & -\xi & 0\\ 0 & 0 & -\xi\end{bmatrix}$	[3]
Continuous	$\infty\infty$	$\begin{bmatrix}\varepsilon & 0 & 0\\ 0 & \varepsilon & 0\\ 0 & 0 & \varepsilon\end{bmatrix}$	$\begin{bmatrix}\mu & 0 & 0\\ 0 & \mu & 0\\ 0 & 0 & \mu\end{bmatrix}$	$\begin{bmatrix}\xi & 0 & 0\\ 0 & \xi & 0\\ 0 & 0 & \xi\end{bmatrix}$	$\begin{bmatrix}\zeta & 0 & 0\\ 0 & \zeta & 0\\ 0 & 0 & \zeta\end{bmatrix}$	[4]
Continuous	$\infty\infty m1'$	$\begin{bmatrix}\varepsilon & 0 & 0\\ 0 & \varepsilon & 0\\ 0 & 0 & \varepsilon\end{bmatrix}$	$\begin{bmatrix}\mu & 0 & 0\\ 0 & \mu & 0\\ 0 & 0 & \mu\end{bmatrix}$	$\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$	$\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$	[2]
Continuous	$\infty\infty m$	$\begin{bmatrix}\varepsilon & 0 & 0\\ 0 & \varepsilon & 0\\ 0 & 0 & \varepsilon\end{bmatrix}$	$\begin{bmatrix}\mu & 0 & 0\\ 0 & \mu & 0\\ 0 & 0 & \mu\end{bmatrix}$	$\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$	$\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$	[2]
Continuous	$\infty\infty m'$	$\begin{bmatrix}\varepsilon & 0 & 0\\ 0 & \varepsilon & 0\\ 0 & 0 & \varepsilon\end{bmatrix}$	$\begin{bmatrix}\mu & 0 & 0\\ 0 & \mu & 0\\ 0 & 0 & \mu\end{bmatrix}$	$\begin{bmatrix}\xi & 0 & 0\\ 0 & \xi & 0\\ 0 & 0 & \xi\end{bmatrix}$	$\begin{bmatrix}\xi & 0 & 0\\ 0 & \xi & 0\\ 0 & 0 & \xi\end{bmatrix}$	[3]

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Appendix – Ferro/Anti-Ferromagnetic Point Groups

○ Ferromagnetic (31)
○ Anti-ferromagnetic (59)

DISCRETE POINT GROUPS

Crystal Family (# of classes)	Nonmagnetic: Type I $P+P'$	Magnetic: Type II P	Magnetic: Type III $P(H)=H+(P-H)'$				
Triclinic (5)	$11'$	1					
	$\bar{1}1'$	$\bar{1}$	$\bar{1}'$				
Monoclinic (11)	$21'$	2	$2'$				
	$m1'$	m	m'				
	$2/m1'$	$2/m$	$2'/m'$	$2/m'$	$2'/m$		
Orthorhombic (12)	$2221'$	222	$2'2'2$				
	$mm21'$	$mm2$	$m'm'2$	$2'm'm$			
	$mmm1'$	mmm	$m'm'm'$	$m'm'm$	mmm'		
Tetragonal (31)	$41'$	4	$4'$				
	$\bar{4}1'$	$\bar{4}$	$\bar{4}'$				
	$4/m1'$	$4/m$	$4'/m$	$4/m'$	$4'/m'$		
	$4221'$	422	$4'22'$	$42'2'$			
	$4mm1'$	$4mm$	$4'mm'$	$4m'm'$			
	$\bar{4}2m1'$	$\bar{4}2m$	$\bar{4}'2m'$	$\bar{4}'m2'$	$\bar{4}2'm'$		
	$4/mmm1'$	$4/mmm$	$4'/mmm'$	$4/mm'm'$	$4/m'm'm'$	$4/m'mm$	$4'/m'm'm$
Trigonal (16)	$31'$	3					
	$\bar{3}1'$	$\bar{3}$	$\bar{3}'$				
	$321'$	32	$32'$				
	$3m1'$	$3m$	$3m'$				
	$\bar{3}m1'$	$\bar{3}m$	$\bar{3}m'$	$\bar{3}'m'$	$\bar{3}'m$		
Hexagonal (31)	$61'$	6	$6'$				
	$\bar{6}1'$	$\bar{6}$	$\bar{6}'$				
	$6/m1'$	$6/m$	$6'/m'$	$6/m'$	$6'/m$		
	$6221'$	622	$6'22'$	$62'2'$			
	$6mm1'$	$6mm$	$6'mm'$	$6m'm'$			
	$\bar{6}m21'$	$\bar{6}m2$	$\bar{6}'2m'$	$\bar{6}'m'2$	$\bar{6}m'2'$		
	$6/mmm1'$	$6/mmm$	$6'/m'mm'$	$6/mm'm'$	$6/m'm'm'$	$6/m'mm$	$6'/mmm'$
Cubic (16)	$231'$	23					
	$m\bar{3}1'$	$m\bar{3}$	$m'\bar{3}'$				
	$4321'$	432	$4'32'$				
	$\bar{4}3m1'$	$\bar{4}3m$	$\bar{4}'3m'$				
	$m\bar{3}m1'$	$m\bar{3}m$	$m'\bar{3}m'$	$m\bar{3}'m'$	$m'\bar{3}'m$		
	32	+ 32	+ 58	= 122 discrete groups			

CONTINUOUS POINT GROUPS

Nonmagnetic: Type I $P+P'$	Magnetic: Type II P	Magnetic: Type III $P(H)=H+(P-H)'$		
$\infty 1'$	∞			
$\infty/m1'$	∞/m	∞/m'		
$\infty 21'$	$\infty 2$	$\infty 2'$		
$\infty m1'$	∞m	$\infty m'$		
$\infty/mm1'$	∞/mm	∞/mm'	$\infty/m'm$	$\infty/m'm'$
$\infty\infty 1'$	$\infty\infty$			
$\infty\infty m1'$	$\infty\infty m$	$\infty\infty m'$		
7	+ 7	+ 7	= 21 continuous groups	

SPATIAL SYMMETRIES : Mirror, Rotation, Inversion through origin
TEMPORAL SYMMETRIES : Time Inversion (denoted by prime)

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Note1: P' = Time – reversed point group symmetry elements

Note2: P = Point group symmetry elements

Note3: H = Halving point subgroup of P

Appendix – Discrete Mirror Symmetries

$$m_x=\begin{bmatrix}-1&0&0\\0&1&0\\0&0&1\end{bmatrix},\ m_y=\begin{bmatrix}1&0&0\\0&-1&0\\0&0&1\end{bmatrix},\ m_z=\begin{bmatrix}1&0&0\\0&1&0\\0&0&-1\end{bmatrix},\ m_{xy}=\begin{bmatrix}0&-1&0\\-1&0&0\\0&0&1\end{bmatrix},\ m_{\bar{x}y}=\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix},\ m_{x\sqrt{3}y}=\begin{bmatrix}\frac{1}{2}&-\frac{\sqrt{3}}{2}&0\\-\frac{\sqrt{3}}{2}&-\frac{1}{2}&0\\0&0&1\end{bmatrix},\ m_{\bar{x}\sqrt{3}y}=\begin{bmatrix}\frac{1}{2}&\frac{\sqrt{3}}{2}&0\\\frac{\sqrt{3}}{2}&-\frac{1}{2}&0\\0&0&1\end{bmatrix}$$

$$m_{\sqrt{3}xy}=\begin{bmatrix}-\frac{1}{2}&-\frac{\sqrt{3}}{2}&0\\-\frac{\sqrt{3}}{2}&\frac{1}{2}&0\\0&0&1\end{bmatrix},\ m_{\sqrt{3}x\bar{y}}=\begin{bmatrix}-\frac{1}{2}&\frac{\sqrt{3}}{2}&0\\\frac{\sqrt{3}}{2}&\frac{1}{2}&0\\0&0&1\end{bmatrix},\ m_{xz}=\begin{bmatrix}0&0&-1\\0&1&0\\-1&0&0\end{bmatrix},\ m_{x\bar{z}}=\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix},\ m_{yz}=\begin{bmatrix}1&0&0\\0&0&-1\\0&-1&0\end{bmatrix},\ m_{y\bar{z}}=\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}$$

Appendix – Discrete Rotational Symmetries

One – fold rotation axis of symmetry

$$1=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

Notes :

(1) *Rotation is in right – hand rule sense with thumb pointing out from the origin along symmetry axis*
Example : 4_z = *rotation by* $+90°$ *about* z *– axis in right – hand sense (i.e., counter – clockwise)*
-4_z = *rotation by* $-90°$ *about* z *– axis in right – hand sense (i.e., clockwise)*

(2) n *– fold axis is symmetry rotation by* $\theta = 360°/n$

(3) *Primed operators (e.g.,* $1', 4'_z$*, etc.) are obtained by including time inversion*

Two – fold rotation axis of symmetry

$$2_x=\begin{bmatrix}1&0&0\\0&-1&0\\0&0&-1\end{bmatrix},\ 2_y=\begin{bmatrix}-1&0&0\\0&1&0\\0&0&-1\end{bmatrix},\ 2_z=\begin{bmatrix}-1&0&0\\0&-1&0\\0&0&1\end{bmatrix},\ 2_{xy}=\begin{bmatrix}0&1&0\\1&0&0\\0&0&-1\end{bmatrix},\ 2_{\bar{x}y}=\begin{bmatrix}0&-1&0\\-1&0&0\\0&0&-1\end{bmatrix},\ 2_{x\sqrt{3}y}=\begin{bmatrix}-\frac{1}{2}&\frac{\sqrt{3}}{2}&0\\\frac{\sqrt{3}}{2}&\frac{1}{2}&0\\0&0&-1\end{bmatrix},\ 2_{\bar{x}\sqrt{3}y}=\begin{bmatrix}-\frac{1}{2}&-\frac{\sqrt{3}}{2}&0\\-\frac{\sqrt{3}}{2}&\frac{1}{2}&0\\0&0&-1\end{bmatrix}$$

$$2_{\sqrt{3}xy}=\begin{bmatrix}\frac{1}{2}&\frac{\sqrt{3}}{2}&0\\\frac{\sqrt{3}}{2}&-\frac{1}{2}&0\\0&0&-1\end{bmatrix},\ 2_{\sqrt{3}x\bar{y}}=\begin{bmatrix}\frac{1}{2}&-\frac{\sqrt{3}}{2}&0\\-\frac{\sqrt{3}}{2}&-\frac{1}{2}&0\\0&0&-1\end{bmatrix},\ 2_{xz}=\begin{bmatrix}0&0&1\\0&-1&0\\1&0&0\end{bmatrix},\ 2_{\bar{x}z}=\begin{bmatrix}0&0&-1\\0&-1&0\\-1&0&0\end{bmatrix},\ 2_{yz}=\begin{bmatrix}-1&0&0\\0&0&1\\0&1&0\end{bmatrix},\ 2_{\bar{y}z}=\begin{bmatrix}-1&0&0\\0&0&-1\\0&-1&0\end{bmatrix}$$

Three – fold rotation axis of symmetry

$$3_z=\begin{bmatrix}-\frac{1}{2}&\frac{\sqrt{3}}{2}&0\\-\frac{\sqrt{3}}{2}&-\frac{1}{2}&0\\0&0&1\end{bmatrix},\ -3_z=\begin{bmatrix}-\frac{1}{2}&-\frac{\sqrt{3}}{2}&0\\\frac{\sqrt{3}}{2}&-\frac{1}{2}&0\\0&0&1\end{bmatrix},\ 3_{xyz}=\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix},\ -3_{xyz}=\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix},\ 3_{\bar{x}yz}=\begin{bmatrix}0&0&-1\\-1&0&0\\0&1&0\end{bmatrix},\ -3_{\bar{x}yz}=\begin{bmatrix}0&-1&0\\0&0&1\\-1&0&0\end{bmatrix},\ 3_{x\bar{y}z}=\begin{bmatrix}0&0&1\\-1&0&0\\0&-1&0\end{bmatrix}$$

$$-3_{x\bar{y}z}=\begin{bmatrix}0&-1&0\\0&0&-1\\1&0&0\end{bmatrix},\ 3_{xy\bar{z}}=\begin{bmatrix}0&0&-1\\1&0&0\\0&-1&0\end{bmatrix},\ -3_{xy\bar{z}}=\begin{bmatrix}0&1&0\\0&0&-1\\-1&0&0\end{bmatrix}$$

Four – fold rotation axis of symmetry

$$4_x=\begin{bmatrix}1&0&0\\0&0&1\\0&-1&0\end{bmatrix},\ -4_x=\begin{bmatrix}1&0&0\\0&0&-1\\0&1&0\end{bmatrix},\ 4_y=\begin{bmatrix}0&0&-1\\0&1&0\\1&0&0\end{bmatrix},\ -4_y=\begin{bmatrix}0&0&1\\0&1&0\\-1&0&0\end{bmatrix},\ 4_z=\begin{bmatrix}0&1&0\\-1&0&0\\0&0&1\end{bmatrix},\ -4_z=\begin{bmatrix}0&-1&0\\1&0&0\\0&0&1\end{bmatrix}$$

Six – fold rotation axis of symmetry

$$6_z=\begin{bmatrix}\frac{1}{2}&\frac{\sqrt{3}}{2}&0\\-\frac{\sqrt{3}}{2}&\frac{1}{2}&0\\0&0&1\end{bmatrix},\ -6_z=\begin{bmatrix}\frac{1}{2}&-\frac{\sqrt{3}}{2}&0\\\frac{\sqrt{3}}{2}&\frac{1}{2}&0\\0&0&1\end{bmatrix}$$

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Appendix – Discrete Roto-Inversion Symmetries

One – fold roto – inversion axis of symmetry

$$\bar{1}=-1=\begin{bmatrix}-1&0&0\\0&-1&0\\0&0&-1\end{bmatrix}$$

Notes :
Example : $\bar{3}_z=(\bar{1})(3_z)$, $-\bar{3}_z=(\bar{1})(-3_z)$

Three – fold roto – inversion axis of symmetry

$$\bar{3}_z=\begin{bmatrix}\frac{1}{2}&-\frac{\sqrt{3}}{2}&0\\\frac{\sqrt{3}}{2}&\frac{1}{2}&0\\0&0&-1\end{bmatrix},\ -\bar{3}_z=\begin{bmatrix}\frac{1}{2}&\frac{\sqrt{3}}{2}&0\\-\frac{\sqrt{3}}{2}&\frac{1}{2}&0\\0&0&-1\end{bmatrix},\ \bar{3}_{xyz}=\begin{bmatrix}0&-1&0\\0&0&-1\\-1&0&0\end{bmatrix},\ -\bar{3}_{xyz}=\begin{bmatrix}0&0&-1\\-1&0&0\\0&-1&0\end{bmatrix},\ \bar{3}_{\bar{x}yz}=\begin{bmatrix}0&0&1\\1&0&0\\0&-1&0\end{bmatrix},\ -\bar{3}_{\bar{x}yz}=\begin{bmatrix}0&1&0\\0&0&-1\\1&0&0\end{bmatrix},\ \bar{3}_{x\bar{y}z}=\begin{bmatrix}0&0&-1\\1&0&0\\0&1&0\end{bmatrix}$$

$$-\bar{3}_{x\bar{y}z}=\begin{bmatrix}0&1&0\\0&0&1\\-1&0&0\end{bmatrix},\ \bar{3}_{xy\bar{z}}=\begin{bmatrix}0&0&1\\-1&0&0\\0&1&0\end{bmatrix},\ -\bar{3}_{xy\bar{z}}=\begin{bmatrix}0&-1&0\\0&0&1\\1&0&0\end{bmatrix}$$

Four – fold roto – inversion axis of symmetry

$$\bar{4}_x=\begin{bmatrix}-1&0&0\\0&0&-1\\0&1&0\end{bmatrix},\ -\bar{4}_x=\begin{bmatrix}-1&0&0\\0&0&1\\0&-1&0\end{bmatrix},\ \bar{4}_y=\begin{bmatrix}0&0&1\\0&-1&0\\-1&0&0\end{bmatrix},\ -\bar{4}_y=\begin{bmatrix}0&0&-1\\0&-1&0\\1&0&0\end{bmatrix},\ \bar{4}_z=\begin{bmatrix}0&-1&0\\1&0&0\\0&0&-1\end{bmatrix},\ -\bar{4}_z=\begin{bmatrix}0&1&0\\-1&0&0\\0&0&-1\end{bmatrix}$$

Six – fold roto – inversion axis of symmetry

$$\bar{6}_z=\begin{bmatrix}-\frac{1}{2}&-\frac{\sqrt{3}}{2}&0\\\frac{\sqrt{3}}{2}&-\frac{1}{2}&0\\0&0&-1\end{bmatrix},\ -\bar{6}_z=\begin{bmatrix}-\frac{1}{2}&\frac{\sqrt{3}}{2}&0\\-\frac{\sqrt{3}}{2}&-\frac{1}{2}&0\\0&0&-1\end{bmatrix}$$

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Appendix – Continuous Symmetries

Continuous mirror – plane axis of symmetry

$$m_{xy\infty} = \begin{bmatrix} -\cos 2\theta & -\sin 2\theta & 0 \\ -\sin 2\theta & \cos 2\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Continuous two – fold rotation axis of symmetry

$$2_{xy\infty} = \begin{bmatrix} \cos 2\theta & \sin 2\theta & 0 \\ \sin 2\theta & -\cos 2\theta & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Continuous rotation axis of symmetry

$$\infty_z = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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2019 International Symposium on Electromagnetic Theory

LECTURE #3
Field and Potential-Based Methods of Analysis

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Field and Potential-Based Methods of Analysis - Overview

COMPUTATIONAL
- FDTD
- MoM
- FEM

FIELD DECOMPOSITION
- TE,TM,TEM
- Transverse/Longitudinal
- Mode Matching
- Use of Symmetry/Invariance

DIFFERENTIAL EQUATION (ANALYTICAL)
- Separation of Variables
- Method of Undetermined Coefficients
- Green's Function

EQUIVALENCE/EM THEOREMS
- Love's Equivalence
- Physical Equivalence
- Volume Equivalence
- Image Theory
- Duality
- Lorentz Reciprocity

APPROXIMATE
- Far-Field Analysis
- Perturbational/Variational
- Asymptotic Analysis
- Born Appoximation
- GO,GTD,UTD (Ray Based)
- PO,PTD,ILDC (Current Based)

TRANSFORM
- Phasor Domain
- Fourier Series
- Fourier Transform
- Laplace Transform
- Complex-plane Analysis

POTENTIALS
- Scalar Potentials
- Vector Potentials

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Factors That Influence Analysis Method – Overview

FACTORS THAT INFLUENCE CHOICE OF ANALYSIS METHOD

1. **Constitutive relations/material tensor form.**
2. **Sources vs. Source-free.**
3. **Field type and invariance.**
4. **Radiation/Scattering/Propagation environment.**
5. **Complexity of problem (CEM often needed)**

EM theorems can aid in analysis – and are reviewed next.

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Fundamental Theorems – Duality

$$\nabla \times \vec{E} = -\vec{J}_h - j\omega\vec{B} \qquad \nabla \cdot \vec{D} = \rho_e \qquad \vec{D} = \overleftrightarrow{\varepsilon} \cdot \vec{E} + \overleftrightarrow{\xi} \cdot \vec{H}$$

$$\nabla \times \vec{H} = \vec{J}_e + j\omega\vec{D} \qquad \nabla \cdot \vec{B} = \rho_h \qquad \vec{B} = \overleftrightarrow{\zeta} \cdot \vec{E} + \overleftrightarrow{\mu} \cdot \vec{H}$$

$$\begin{aligned} \vec{E} &\leftrightarrow \vec{H} \\ \vec{D} &\leftrightarrow -\vec{B} \\ \rho_e &\leftrightarrow -\rho_h \\ \vec{J}_e &\leftrightarrow -\vec{J}_h \\ \overleftrightarrow{\varepsilon} &\leftrightarrow -\overleftrightarrow{\mu} \\ \overleftrightarrow{\xi} &\leftrightarrow -\overleftrightarrow{\zeta} \end{aligned} \quad \cdots \quad \textit{a duality transformation (not unique)}$$

If invoking duality, make sure any boundary conditions are also dual to the original boundary conditions (e.g., PMC dual to PEC)!

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Fundamental Theorems – Reciprocity

$$\nabla\times\vec{E}_{1,2}=-\vec{J}_{h1,2}-j\omega\vec{B}_{1,2},\quad \vec{D}_{1,2}=\vec{\vec{\varepsilon}}\cdot\vec{E}_{1,2}+\vec{\vec{\xi}}\cdot\vec{H}_{1,2}$$
$$\nabla\times\vec{H}_{1,2}=\vec{J}_{e1,2}+j\omega\vec{D}_{1,2},\quad \vec{B}_{1,2}=\vec{\vec{\zeta}}\cdot\vec{E}_{1,2}+\vec{\vec{\mu}}\cdot\vec{H}_{1,2}\quad \ldots$$

two different sources / fields, but same frequency and medium

$$\nabla\cdot(\vec{E}_1\times\vec{H}_2-\vec{E}_2\times\vec{H}_1)=(\vec{E}_2\cdot\vec{J}_{e1}-\vec{E}_1\cdot\vec{J}_{e2}-\vec{H}_2\cdot\vec{J}_{h1}+\vec{H}_1\cdot\vec{J}_{h2})+j\omega\underbrace{(\vec{H}_1\cdot\vec{B}_2-\vec{H}_2\cdot\vec{B}_1-\vec{E}_1\cdot\vec{D}_2+\vec{E}_2\cdot\vec{D}_1)}_{\text{if }=0\ \Rightarrow\ \text{reciprocal environment } 1\leftrightarrow 2}$$

(source and observer can be interchanged)

Note: $\nabla\cdot(\vec{A}\times\vec{B})=\vec{B}\cdot\nabla\times\vec{A}-\vec{A}\cdot\nabla\times\vec{B}$

$$\vec{H}_1\cdot\vec{B}_2-\vec{H}_2\cdot\vec{B}_1-\vec{E}_1\cdot\vec{D}_2+\vec{E}_2\cdot\vec{D}_1=0\ \ldots\text{if}\ \begin{matrix}\vec{E}_1\cdot\vec{\vec{\varepsilon}}\cdot\vec{E}_2=\vec{E}_2\cdot\vec{\vec{\varepsilon}}\cdot\vec{E}_1\\ \vec{H}_1\cdot\vec{\vec{\mu}}\cdot\vec{H}_2=\vec{H}_2\cdot\vec{\vec{\mu}}\cdot\vec{H}_1\\ \vec{E}_1\cdot\vec{\vec{\xi}}\cdot\vec{H}_2=-\vec{H}_2\cdot\vec{\vec{\zeta}}\cdot\vec{E}_1\\ \vec{H}_1\cdot\vec{\vec{\zeta}}\cdot\vec{E}_2=-\vec{E}_2\cdot\vec{\vec{\xi}}\cdot\vec{H}_1\end{matrix}\ \Rightarrow\ \begin{matrix}\vec{\vec{\varepsilon}}=\vec{\vec{\varepsilon}}^T\\ \vec{\vec{\mu}}=\vec{\vec{\mu}}^T\\ \vec{\vec{\xi}}=-\vec{\vec{\zeta}}^T\\ \vec{\vec{\zeta}}=-\vec{\vec{\xi}}^T\end{matrix}\ \ldots\text{for reciprocal media}$$

Much research occurring in non-reciprocal media – especially at optical frequencies!

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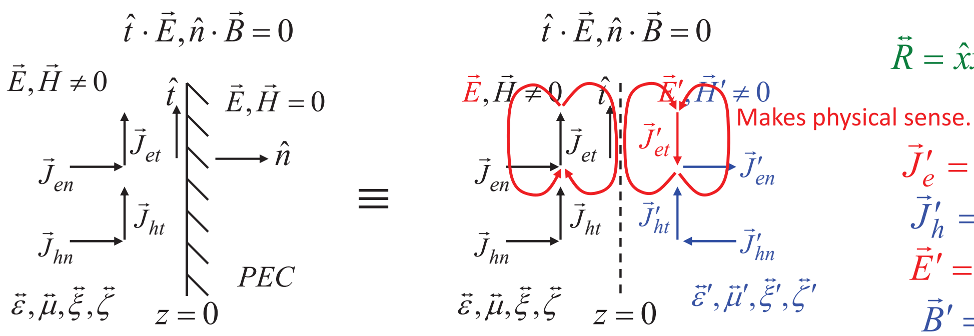
Fundamental Theorems – Image Theory



$$\vec{\vec{R}}=\hat{x}\hat{x}+\hat{y}\hat{y}-\hat{z}\hat{z}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&-1\end{bmatrix}$$

$$\begin{matrix}\vec{J}'_e=-\vec{\vec{R}}\cdot\vec{J}_e\\ \vec{J}'_h=\vec{\vec{R}}\cdot\vec{J}_h\\ \vec{E}'=-\vec{\vec{R}}\cdot\vec{E}\\ \vec{B}'=\vec{\vec{R}}\cdot\vec{B}\end{matrix}\ \ldots\ \text{to ensure}\ \hat{t}\cdot\vec{E},\hat{n}\cdot\vec{B}=0$$

$$\begin{matrix}\vec{\vec{R}}\cdot c\vec{D}=\vec{\vec{R}}\cdot\vec{\vec{P}}\cdot\vec{E}+\vec{\vec{R}}\cdot\vec{\vec{L}}\cdot c\vec{B}\\ \vec{\vec{R}}\cdot\vec{H}=\vec{\vec{R}}\cdot\vec{\vec{M}}\cdot\vec{E}+\vec{\vec{R}}\cdot\vec{\vec{Q}}\cdot c\vec{B}\end{matrix}\ \Rightarrow\ \begin{matrix}\overbrace{-\vec{\vec{R}}\cdot c\vec{D}}^{c\vec{D}'}=\overbrace{\vec{\vec{R}}\cdot\vec{\vec{P}}\cdot\vec{\vec{R}}}^{\vec{\vec{P}}'}\cdot\overbrace{(-\vec{\vec{R}}\cdot\vec{E})}^{\vec{E}'}+\overbrace{-\vec{\vec{R}}\cdot\vec{\vec{L}}\cdot\vec{\vec{R}}}^{\vec{\vec{L}}'}\cdot\overbrace{\vec{\vec{R}}\cdot c\vec{B}}^{c\vec{B}'}\\ \underbrace{\vec{\vec{R}}\cdot\vec{H}}_{\vec{H}'}=\underbrace{-\vec{\vec{R}}\cdot\vec{\vec{M}}\cdot\vec{\vec{R}}}_{\vec{\vec{M}}'}\cdot\underbrace{(-\vec{\vec{R}}\cdot\vec{E})}_{\vec{E}'}+\underbrace{\vec{\vec{R}}\cdot\vec{\vec{Q}}\cdot\vec{\vec{R}}}_{\vec{\vec{Q}}'}\cdot\underbrace{\vec{\vec{R}}\cdot c\vec{B}}_{c\vec{B}'}\end{matrix}\ \ldots$$

$\vec{\vec{C}}_{EB}$ formulation (reveals how $\vec{D}',\vec{H}',\vec{\vec{P}}',\vec{\vec{L}}',\vec{\vec{M}}',\vec{\vec{Q}}'$ transform)

J. Kong, "Image Theory for Bianisotropic Media," IEEE Trans. Ant. Prop., May 1971.

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Fundamental Theorems – Image Theory

$$\begin{aligned}\overleftrightarrow{R}\cdot\vec{D} &= \overleftrightarrow{R}\cdot\overleftrightarrow{\varepsilon}\cdot\vec{E}+\overleftrightarrow{R}\cdot\overleftrightarrow{\xi}\cdot\vec{H}\\ \overleftrightarrow{R}\cdot\vec{B} &= \overleftrightarrow{R}\cdot\overleftrightarrow{\zeta}\cdot\vec{E}+\overleftrightarrow{R}\cdot\overleftrightarrow{\mu}\cdot\vec{H}\end{aligned} \Rightarrow \begin{aligned}\overbrace{-\overleftrightarrow{R}\cdot\vec{D}}^{\vec{D}'} &= \overbrace{\overleftrightarrow{R}\cdot\overleftrightarrow{\varepsilon}\cdot\overleftrightarrow{R}}^{\overleftrightarrow{\varepsilon}'}\cdot\overbrace{(-\overleftrightarrow{R}\cdot\vec{E})}^{\vec{E}'}+\overbrace{-\overleftrightarrow{R}\cdot\overleftrightarrow{\xi}\cdot\overleftrightarrow{R}}^{\overleftrightarrow{\xi}'}\cdot\overbrace{\overleftrightarrow{R}\cdot\vec{H}}^{\vec{H}'}\\ \underbrace{\overleftrightarrow{R}\cdot\vec{B}}_{\vec{B}'} &= \underbrace{-\overleftrightarrow{R}\cdot\overleftrightarrow{\zeta}\cdot\overleftrightarrow{R}}_{\overleftrightarrow{\zeta}'}\cdot\underbrace{(-\overleftrightarrow{R}\cdot\vec{E})}_{\vec{E}'}+\underbrace{\overleftrightarrow{R}\cdot\overleftrightarrow{\mu}\cdot\overleftrightarrow{R}}_{\overleftrightarrow{\mu}'}\cdot\underbrace{\overleftrightarrow{R}\cdot\vec{H}}_{\vec{H}'}\end{aligned} \ldots\overleftrightarrow{C}_{EH}\ \mathit{formulation}$$

$$\overleftrightarrow{\varepsilon}'=\overleftrightarrow{R}\cdot\overleftrightarrow{\varepsilon}\cdot\overleftrightarrow{R}=\begin{bmatrix}\varepsilon_{xx} & \varepsilon_{xy} & -\varepsilon_{xz}\\ \varepsilon_{yx} & \varepsilon_{yy} & -\varepsilon_{yz}\\ -\varepsilon_{zx} & -\varepsilon_{zy} & \varepsilon_{zz}\end{bmatrix},\quad \overleftrightarrow{\mu}'=\overleftrightarrow{R}\cdot\overleftrightarrow{\mu}\cdot\overleftrightarrow{R}=\begin{bmatrix}\mu_{xx} & \mu_{xy} & -\mu_{xz}\\ \mu_{yx} & \mu_{yy} & -\mu_{yz}\\ -\mu_{zx} & -\mu_{zy} & \mu_{zz}\end{bmatrix}$$

$$\overleftrightarrow{\xi}'=-\overleftrightarrow{R}\cdot\overleftrightarrow{\xi}\cdot\overleftrightarrow{R}=\begin{bmatrix}-\xi_{xx} & -\xi_{xy} & \xi_{xz}\\ -\xi_{yx} & -\xi_{yy} & \xi_{yz}\\ \xi_{zx} & \xi_{zy} & -\xi_{zz}\end{bmatrix},\quad \overleftrightarrow{\zeta}'=-\overleftrightarrow{R}\cdot\overleftrightarrow{\zeta}\cdot\overleftrightarrow{R}=\begin{bmatrix}-\zeta_{xx} & -\zeta_{xy} & \zeta_{xz}\\ -\zeta_{yx} & -\zeta_{yy} & \zeta_{yz}\\ \zeta_{zx} & \zeta_{zy} & -\zeta_{zz}\end{bmatrix}$$

$$\ldots\ \underbrace{\mathit{image\ medium}}_{\mathit{what\ media\ is\ image\ theory\ most\ amenable?}}$$

J. Kong, “Image Theory for Bianisotropic Media,” IEEE Trans. Ant. Prop., May 1971.

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Field Based Analysis – SIMPLE Media

$$\begin{aligned}\nabla\times\vec{E} &= -\vec{J}_h - j\omega\mu\vec{H}\ \ (1)\\ \nabla\times\vec{H} &= \vec{J}_e + j\omega\varepsilon\vec{E}\ \ (2)\end{aligned}$$

$$(1)\ \Rightarrow\ \boxed{\vec{H}=-\frac{\nabla\times\vec{E}+\vec{J}_h}{j\omega\mu}}\ (3)$$

$$(3)\to(2)\ \Rightarrow\ \boxed{\nabla\times\nabla\times\vec{E}-k^2\vec{E}=-j\omega\mu\vec{J}_e-\nabla\times\vec{J}_h\ ,\ k^2=\omega^2\varepsilon\mu}\ \mathit{or}$$

$$\begin{aligned}\nabla\times\nabla\times\vec{E} &= \nabla(\nabla\cdot\vec{E})-\nabla^2\vec{E}\\ \nabla\cdot\vec{E} &= \frac{\rho_e}{\varepsilon}=-\frac{\nabla\cdot\vec{J}_e}{j\omega\varepsilon}\end{aligned}\ \Rightarrow\ \boxed{\nabla^2\vec{E}+k^2\vec{E}=j\omega\mu\vec{J}_e\underbrace{-\nabla\left(\frac{\nabla\cdot\vec{J}_e}{j\omega\varepsilon}\right)+\nabla\times\vec{J}_h}_{\mathit{not\ so\ fun}}}\ \ldots\ \begin{array}{c}\mathit{well-known}\\ \mathit{result}\end{array}$$

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Field Based Analysis – General Bianisotropic Media

$$\begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix}, \begin{bmatrix} \mu_{xx} & \mu_{xy} & \mu_{xz} \\ \mu_{yx} & \mu_{yy} & \mu_{yz} \\ \mu_{zx} & \mu_{zy} & \mu_{zz} \end{bmatrix}, \begin{bmatrix} \xi_{xx} & \xi_{xy} & \xi_{xz} \\ \xi_{yx} & \xi_{yy} & \xi_{yz} \\ \xi_{zx} & \xi_{zy} & \xi_{zz} \end{bmatrix}, \begin{bmatrix} \zeta_{xx} & \zeta_{xy} & \zeta_{xz} \\ \zeta_{yx} & \zeta_{yy} & \zeta_{yz} \\ \zeta_{zx} & \zeta_{zy} & \zeta_{zz} \end{bmatrix} \Rightarrow \textit{Must use Maxwell's equations directly}$$

$$\nabla\times\vec{E} = \nabla\times\overbrace{\overleftrightarrow{I}\cdot\vec{E}}^{\textit{Key point!}} = -\vec{J}_h - j\omega\overleftrightarrow{\mu}\cdot\vec{H} - j\omega\overleftrightarrow{\zeta}\cdot\vec{E} \quad or \quad (\nabla\times\overleftrightarrow{I} + j\omega\overleftrightarrow{\zeta})\cdot\vec{E} = -\vec{J}_h - j\omega\overleftrightarrow{\mu}\cdot\vec{H} \quad (1)$$

$$\nabla\times\vec{H} = \nabla\times\overleftrightarrow{I}\cdot\vec{H} = \vec{J}_e + j\omega\overleftrightarrow{\varepsilon}\cdot\vec{E} + j\omega\overleftrightarrow{\xi}\cdot\vec{H} \qquad (\nabla\times\overleftrightarrow{I} - j\omega\overleftrightarrow{\xi})\cdot\vec{H} = \vec{J}_e + j\omega\overleftrightarrow{\varepsilon}\cdot\vec{E} \quad (2)$$

$$(1) \Rightarrow \boxed{\vec{H} = -\frac{\overleftrightarrow{\mu}^{-1}\cdot(\nabla\times\overleftrightarrow{I} + j\omega\overleftrightarrow{\zeta})\cdot\vec{E}}{j\omega} - \frac{\overleftrightarrow{\mu}^{-1}\cdot\vec{J}_h}{j\omega}} \quad (3)$$

$(3)\rightarrow(2) \Rightarrow$

$$\boxed{\overleftrightarrow{W}_E\cdot\vec{E} = [(\nabla\times\overleftrightarrow{I} - j\omega\overleftrightarrow{\xi})\cdot\overleftrightarrow{\mu}^{-1}\cdot(\nabla\times\overleftrightarrow{I} + j\omega\overleftrightarrow{\zeta}) - \omega^2\overleftrightarrow{\varepsilon}]\cdot\vec{E} = -j\omega\vec{J}_e - (\nabla\times\overleftrightarrow{I} - j\omega\overleftrightarrow{\xi})\cdot\overleftrightarrow{\mu}^{-1}\cdot\vec{J}_h}$$

$$\boxed{[\overleftrightarrow{W}_E \textit{ is } 3\times3]}$$

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Field Based Analysis – Specialized Bianisotropic Media (Transverse/Longitudinal Decomposition)

$$\overleftrightarrow{\kappa} = \begin{bmatrix} \kappa_{xx} & \kappa_{xy} & 0 \\ \kappa_{yx} & \kappa_{yy} & 0 \\ 0 & 0 & \kappa_z \end{bmatrix} = \overleftrightarrow{\kappa}_t + \hat{z}\kappa_z\hat{z} \quad \ldots for\ \overleftrightarrow{\kappa} = \overleftrightarrow{\varepsilon}, \overleftrightarrow{\mu}, \overleftrightarrow{\xi}, \overleftrightarrow{\zeta} \Rightarrow$$

2×2, 1×1

$$\overbrace{\begin{aligned} \nabla_t\times\hat{z}H_z + \hat{z}\frac{\partial}{\partial z}\times\vec{H}_t &= \vec{J}_{et} + j\omega\overleftrightarrow{\varepsilon}_t\cdot\vec{E}_t + j\omega\overleftrightarrow{\xi}_t\cdot\vec{H}_t \\ \nabla_t\times\hat{z}E_z + \hat{z}\frac{\partial}{\partial z}\times\vec{E}_t &= -\vec{J}_{ht} - j\omega\overleftrightarrow{\mu}_t\cdot\vec{H}_t - j\omega\overleftrightarrow{\zeta}_t\cdot\vec{E}_t \end{aligned}}^{\textit{Transverse Relations}}$$

$$\overbrace{\begin{aligned} \nabla_t\times\vec{H}_t &= \hat{z}J_{ez} + \hat{z}j\omega\varepsilon_z E_z + \hat{z}j\omega\xi_z H_z \\ \nabla_t\times\vec{E}_t &= -\hat{z}J_{hz} - \hat{z}j\omega\mu_z H_z - \hat{z}j\omega\zeta_z E_z \end{aligned}}^{\textit{Longitudinal Relations}} \quad or \quad \begin{bmatrix} \hat{z}E_z \\ \hat{z}H_z \end{bmatrix} = \frac{1}{j\omega\underbrace{(\varepsilon_z\mu_z - \xi_z\zeta_z)}_{\Delta_z}} \begin{bmatrix} \mu_z & -\xi_z \\ -\zeta_z & \varepsilon_z \end{bmatrix} \begin{bmatrix} \nabla_t\times\vec{H}_t - \hat{z}J_{ez} \\ -\nabla_t\times\vec{E}_t - \hat{z}J_{hz} \end{bmatrix}$$

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Field Based Analysis – Specialized Bianisotropic Media (Transverse/Longitudinal Decomposition)

$$\boxed{\underline{\overleftrightarrow{L}}_t \cdot \underline{\vec{F}}_t = \underline{\overleftrightarrow{L}}_{st} \cdot \underline{\vec{J}} \;\Rightarrow\; \underline{\vec{F}}_t = \underline{\overleftrightarrow{L}}_t^{-1} \cdot \underline{\overleftrightarrow{L}}_{st} \cdot \underline{\vec{J}}} \qquad [\underline{\overleftrightarrow{L}}_t \text{ is block } 2\times 2]$$

$$\underline{\overleftrightarrow{L}}_t = \begin{bmatrix} \frac{\varepsilon_z}{\Delta_z}\overleftrightarrow{\mu}_t \cdot \nabla_t \times \nabla_t \times \overleftrightarrow{I}_t - \omega^2 \overleftrightarrow{\mu}_t \cdot \overleftrightarrow{\varepsilon}_t & \frac{\zeta_z}{\Delta_z}\overleftrightarrow{\mu}_t \cdot \nabla_t \times \nabla_t \times \overleftrightarrow{I}_t - j\omega\overleftrightarrow{\mu}_t \cdot \hat{z}\frac{\partial}{\partial z} \times \overleftrightarrow{I}_t - \omega^2 \overleftrightarrow{\mu}_t \cdot \overleftrightarrow{\xi}_t \\ \frac{\xi_z}{\Delta_z}\overleftrightarrow{\varepsilon}_t \cdot \nabla_t \times \nabla_t \times \overleftrightarrow{I}_t + j\omega\overleftrightarrow{\varepsilon}_t \cdot \hat{z}\frac{\partial}{\partial z} \times \overleftrightarrow{I}_t - \omega^2 \overleftrightarrow{\varepsilon}_t \cdot \overleftrightarrow{\zeta}_t & \frac{\mu_z}{\Delta_z}\overleftrightarrow{\varepsilon}_t \cdot \nabla_t \times \nabla_t \times \overleftrightarrow{I}_t - \omega^2 \overleftrightarrow{\varepsilon}_t \cdot \overleftrightarrow{\mu}_t \end{bmatrix}$$

$$\underline{\overleftrightarrow{L}}_{st} = \begin{bmatrix} -j\omega\overleftrightarrow{\mu}_t + \frac{\zeta_z}{\Delta_z}\overleftrightarrow{\mu}_t \cdot \nabla_t \times \hat{z}\hat{z} & -\frac{\varepsilon_z}{\Delta_z}\overleftrightarrow{\mu}_t \cdot \nabla_t \times \hat{z}\hat{z} \\ \frac{\mu_z}{\Delta_z}\overleftrightarrow{\varepsilon}_t \cdot \nabla_t \times \hat{z}\hat{z} & -j\omega\overleftrightarrow{\varepsilon}_t - \frac{\xi_z}{\Delta_z}\overleftrightarrow{\varepsilon}_t \cdot \nabla_t \times \hat{z}\hat{z} \end{bmatrix}, \quad \underline{\vec{F}}_t = \begin{bmatrix} \vec{E}_t \\ \vec{H}_t \end{bmatrix}, \quad \underline{\vec{J}} = \begin{bmatrix} \vec{J}_e \\ \vec{J}_h \end{bmatrix}$$

$$\begin{bmatrix} \hat{z}E_z \\ \hat{z}H_z \end{bmatrix} = \frac{1}{j\omega\Delta_z}\begin{bmatrix} \mu_z & -\xi_z \\ -\zeta_z & \varepsilon_z \end{bmatrix}\begin{bmatrix} \nabla_t \times \vec{H}_t - \hat{z}J_{ez} \\ -\nabla_t \times \vec{E}_t - \hat{z}J_{hz} \end{bmatrix} \qquad \overleftrightarrow{\kappa} = \begin{bmatrix} \kappa_{xx} & \kappa_{xy} & 0 \\ \kappa_{yx} & \kappa_{yy} & 0 \\ 0 & 0 & \kappa_z \end{bmatrix}$$

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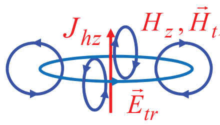
Field Based Analysis – Uniaxial Anisotropic Media (TE/TM Decomposition)

$$\begin{aligned} \nabla \times \vec{H} &= \vec{J}_e + j\omega(\varepsilon_t \overleftrightarrow{I}_t + \hat{z}\hat{z}\varepsilon_z)\cdot\vec{E} \\ \nabla \times \vec{E} &= -\vec{J}_h - j\omega(\mu_t \overleftrightarrow{I}_t + \hat{z}\hat{z}\mu_z)\cdot\vec{H} \end{aligned} \;\Rightarrow\; \overbrace{\begin{aligned} \nabla_t \times \hat{z}H_z + \hat{z}\tfrac{\partial}{\partial z}\times\vec{H}_t &= \vec{J}_{et} + j\omega\varepsilon_t\vec{E}_t \\ \nabla_t \times \hat{z}E_z + \hat{z}\tfrac{\partial}{\partial z}\times\vec{E}_t &= -\vec{J}_{ht} - j\omega\mu_t\vec{H}_t \end{aligned}}^{\textit{Transverse Relations}}, \; \overbrace{\begin{aligned} \nabla \times \vec{H}_t &= \hat{z}J_{ez} + \hat{z}j\omega\varepsilon_z E_z \\ \nabla \times \vec{E}_t &= -\hat{z}J_{hz} - \hat{z}j\omega\mu_z H_z \end{aligned}}^{\textit{Longitudinal Relations}}$$

$$\begin{aligned} \vec{E}_t &= \vec{E}_{t\ell} + \vec{E}_{tr} = \nabla_t\Phi + \nabla_t \times \hat{z}\theta, \quad \vec{J}_{et} = \vec{J}_{et\ell} + \vec{J}_{etr} = \nabla_t u_e + \nabla_t \times \hat{z}v_e \\ \vec{H}_t &= \vec{H}_{t\ell} + \vec{H}_{tr} = \nabla_t\pi + \nabla_t \times \hat{z}\psi, \quad \vec{J}_{ht} = \vec{J}_{ht\ell} + \vec{J}_{htr} = \nabla_t u_h + \nabla_t \times \hat{z}v_h \end{aligned} \quad \cdots \textit{2D Helmholtz expansion.}$$

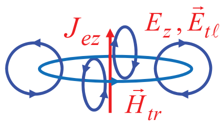
TE^z

$$\nabla_t \times \hat{z}H_z + \hat{z}\times\frac{\partial\vec{H}_{t\ell}}{\partial z} = \vec{J}_{etr} + j\omega\varepsilon_t\vec{E}_{tr}$$
$$\hat{z}\times\frac{\partial\vec{E}_{tr}}{\partial z} = -\vec{J}_{ht\ell} - j\omega\mu_t\vec{H}_{t\ell}$$
$$\nabla_t \times \vec{E}_{tr} = -\hat{z}J_{hz} - \hat{z}j\omega\mu_z H_z$$



$$-\frac{\partial^2\theta}{\partial z^2} - \frac{\mu_t}{\mu_z}\nabla_t^2\theta - k_t^2\theta = -\frac{\mu_t}{\mu_z}J_{hz} + \frac{\partial u_h}{\partial z} - j\omega\mu_t v_e$$
$$\pi = -\frac{1}{j\omega\mu_t}\left(\frac{\partial\theta}{\partial z} + u_h\right), \quad H_z = \frac{1}{j\omega\mu_z}(\nabla_t^2\theta - J_{hz})$$

TM^z



$$\nabla_t \times \hat{z}E_z + \hat{z}\times\frac{\partial\vec{E}_{t\ell}}{\partial z} = -\vec{J}_{htr} - j\omega\mu_t\vec{H}_{tr}$$
$$\hat{z}\times\frac{\partial\vec{H}_{tr}}{\partial z} = \vec{J}_{et\ell} + j\omega\varepsilon_t\vec{E}_{t\ell}$$
$$\nabla_t \times \vec{H}_{tr} = \hat{z}J_{ez} + \hat{z}j\omega\varepsilon_z E_z$$

$$-\frac{\partial^2\psi}{\partial z^2} - \frac{\varepsilon_t}{\varepsilon_z}\nabla_t^2\psi - k_t^2\psi = \frac{\varepsilon_t}{\varepsilon_z}J_{ez} - \frac{\partial u_e}{\partial z} - j\omega\varepsilon_t v_h$$
$$\Phi = \frac{1}{j\omega\varepsilon_t}\left(\frac{\partial\psi}{\partial z} - u_e\right), \quad E_z = -\frac{1}{j\omega\varepsilon_z}(\nabla_t^2\psi + J_{ez})$$

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Field Based Analysis – Anisotropic Gyrotropic Media (y-invariance)

$$\begin{bmatrix}\varepsilon_1 & 0 & j\varepsilon_3\\ 0 & \varepsilon_2 & 0\\ -j\varepsilon_3 & 0 & \varepsilon_1\end{bmatrix},\begin{bmatrix}\mu_1 & 0 & j\mu_3\\ 0 & \mu_2 & 0\\ -j\mu_3 & 0 & \mu_1\end{bmatrix},\ \frac{\partial}{\partial y}=0 \rightarrow \begin{matrix}\nabla\times\vec{H}=\vec{J}_e+j\omega\vec{\vec{\varepsilon}}\cdot\vec{E}\\ \nabla\times\vec{E}=-\vec{J}_h-j\omega\vec{\vec{\mu}}\cdot\vec{H}\end{matrix} \Rightarrow$$

TE^y (TM^x, TM^z)

$$-\frac{\partial H_y}{\partial z}=J_{ex}+j\omega\varepsilon_1 E_x-\omega\varepsilon_3 E_z$$

$$\frac{\partial H_y}{\partial x}=J_{ez}+\omega\varepsilon_3 E_x+j\omega\varepsilon_1 E_z$$

$$-\frac{\partial E_z}{\partial x}+\frac{\partial E_x}{\partial z}=-J_{hy}-j\omega\mu_2 H_y$$

$$\nabla_t^2 H_y+k^2 H_y=-\hat{y}\cdot\nabla_t\times\vec{J}_{et}+j\frac{\varepsilon_3}{\varepsilon_1}\nabla_t\cdot\vec{J}_{et}+j\omega\frac{\varepsilon_1^2-\varepsilon_3^2}{\varepsilon_1}J_{hy}$$

$$\begin{bmatrix}E_x\\E_z\end{bmatrix}=\frac{1}{\omega^2(\varepsilon_1^2-\varepsilon_3^2)}\begin{bmatrix}j\omega\varepsilon_1 & \omega\varepsilon_3\\ -\omega\varepsilon_3 & j\omega\varepsilon_1\end{bmatrix}\begin{bmatrix}J_{ex}+\frac{\partial H_y}{\partial z}\\ J_{ez}-\frac{\partial H_y}{\partial x}\end{bmatrix}$$

$$\nabla_t=\hat{x}\frac{\partial}{\partial x}+\hat{z}\frac{\partial}{\partial z}\ ,\ k^2=\omega^2\frac{(\varepsilon_1^2-\varepsilon_3^2)\mu_2}{\varepsilon_1}$$

TM^y (TE^x, TE^z)

$$-\frac{\partial E_y}{\partial z}=-J_{hx}-j\omega\mu_1 H_x+\omega\mu_3 H_z$$

$$\frac{\partial E_y}{\partial x}=-J_{hz}-\omega\mu_3 H_x-j\omega\mu_1 H_z$$

$$-\frac{\partial H_z}{\partial x}+\frac{\partial H_x}{\partial z}=J_{ey}+j\omega\varepsilon_2 E_y$$

$$\nabla_t^2 E_y+k^2 E_y=\hat{y}\cdot\nabla_t\times\vec{J}_{ht}-j\frac{\mu_3}{\mu_1}\nabla_t\cdot\vec{J}_{ht}+j\omega\frac{\mu_1^2-\mu_3^2}{\mu_1}J_{ey}$$

$$\begin{bmatrix}H_x\\H_z\end{bmatrix}=\frac{1}{\omega^2(\mu_1^2-\mu_3^2)}\begin{bmatrix}j\omega\mu_1 & \omega\mu_3\\ -\omega\mu_3 & j\omega\varepsilon_1\end{bmatrix}\begin{bmatrix}J_{hx}-\frac{\partial E_y}{\partial z}\\ J_{hz}+\frac{\partial E_y}{\partial x}\end{bmatrix}$$

$$\nabla_t=\hat{x}\frac{\partial}{\partial x}+\hat{z}\frac{\partial}{\partial z}\ ,\ k^2=\omega^2\frac{(\mu_1^2-\mu_3^2)\varepsilon_2}{\mu_1}$$

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Vector Potential Based Analysis – SIMPLE Media

$$\nabla\times\vec{E}=-j\omega\mu\vec{H}\quad(1)$$
$$\nabla\times\vec{H}=\vec{J}_e+j\omega\varepsilon\vec{E}\quad(2)$$
$$\nabla\cdot\vec{E}=\frac{\rho_e}{\varepsilon}\quad(3)$$
$$\nabla\cdot\vec{H}=0\ or\ \nabla\cdot\vec{B}=0\quad(4)$$

$$(4)\Rightarrow\ \vec{B}=\mu\vec{H}=\nabla\times\vec{A}\quad(5)$$
$$(5)\rightarrow(1)\Rightarrow\vec{E}=-j\omega\vec{A}-\nabla\Phi_e\quad(6)$$

$$(5),(6)\rightarrow(2)\Rightarrow$$
$$\nabla^2\vec{A}+k^2\vec{A}+\nabla(-\nabla\cdot\vec{A})=-\mu\vec{J}_e+\nabla(j\omega\varepsilon\mu\Phi_e)$$
$$\nabla\cdot\vec{A}=-j\omega\varepsilon\mu\Phi_e\ ...Lorenz\ gauge\Rightarrow$$

$$\boxed{\begin{matrix}\nabla^2\vec{A}+k^2\vec{A}=-\mu\vec{J}_e\\ \vec{H}=\frac{1}{\mu}\nabla\times\vec{A}\\ \vec{E}=\frac{1}{j\omega\varepsilon\mu}(k^2\vec{A}+\nabla\nabla\cdot\vec{A})\end{matrix}}\ ...much\ nicer$$

$$\nabla\times\vec{E}=-\vec{J}_h-j\omega\mu\vec{H}\quad(1)$$
$$\nabla\times\vec{H}=j\omega\varepsilon\vec{E}\quad(2)$$
$$\nabla\cdot\vec{E}=0\ or\ \nabla\cdot\vec{D}=0\quad(3)$$
$$\nabla\cdot\vec{H}=\frac{\rho_h}{\mu}\quad(4)$$

$$(3)\Rightarrow\ \vec{D}=\varepsilon\vec{E}=-\nabla\times\vec{F}\quad(5)$$
$$(5)\rightarrow(2)\Rightarrow\vec{H}=-j\omega\vec{F}-\nabla\Phi_h\quad(6)$$

$$(5),(6)\rightarrow(1)\Rightarrow$$
$$\nabla^2\vec{F}+k^2\vec{F}+\nabla(-\nabla\cdot\vec{F})=-\varepsilon\vec{J}_h+\nabla(j\omega\varepsilon\mu\Phi_h)$$
$$\nabla\cdot\vec{F}=-j\omega\varepsilon\mu\Phi_h\ ...Lorenz\ gauge\Rightarrow$$

$$\boxed{\begin{matrix}\nabla^2\vec{F}+k^2\vec{F}=-\varepsilon\vec{J}_h\\ \vec{E}=-\frac{1}{\varepsilon}\nabla\times\vec{F}\\ \vec{H}=\frac{1}{j\omega\varepsilon\mu}(k^2\vec{F}+\nabla\nabla\cdot\vec{F})\end{matrix}}\ ...much\ nicer$$

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Vector Potential Based Analysis – Anisotropic Media Example/Bad News

$$\nabla\times\vec{E}=-j\omega\vec{\vec{\mu}}\cdot\vec{H}\quad(1)$$
$$\nabla\times\vec{H}=\vec{J}_e+j\omega\vec{\vec{\varepsilon}}\cdot\vec{E}\quad(2)$$
$$\nabla\cdot(\vec{\vec{\varepsilon}}\cdot\vec{E})=\rho_e\quad(3)$$
$$\nabla\cdot\vec{B}=\nabla\cdot(\vec{\vec{\mu}}\cdot\vec{H})=0\quad(4)$$

$$(4)\ \Rightarrow\ \vec{B}=\vec{\vec{\mu}}\cdot\vec{H}=\nabla\times\vec{A}\quad(5)$$
$$(5)\to(1)\Rightarrow\vec{E}=-j\omega\vec{A}-\nabla\Phi_e\quad(6)$$

$$(5),(6)\to(2)\Rightarrow$$
$$\nabla\times(\vec{\vec{\mu}}^{-1}\cdot\nabla\times\vec{A})=\vec{J}_e+\omega^2\vec{\vec{\varepsilon}}\cdot\vec{A}-j\omega\vec{\vec{\varepsilon}}\cdot\nabla\Phi_e$$
$$\nabla\cdot\vec{A}=???\ \Rightarrow$$

Not Isotropic $\Rightarrow$ ~~*Vector Potential*~~

Scalar Potential Based Analysis – Bianisotropic Gyrotropic Media

$$\vec{\vec{\kappa}}=\begin{bmatrix}\kappa_t & -j\kappa_g & 0\\ j\kappa_g & \kappa_t & 0\\ 0 & 0 & \kappa_z\end{bmatrix}=\kappa_t\vec{\vec{I}}_t+j\kappa_g\hat{z}\times\vec{\vec{I}}_t+\hat{z}\kappa_z\hat{z}\ \ldots for\ \vec{\vec{\kappa}}=\vec{\vec{\varepsilon}},\vec{\vec{\mu}},\vec{\vec{\xi}},\vec{\vec{\zeta}}\ \Rightarrow$$

$$\overbrace{\begin{aligned}&\nabla_t\times\hat{z}H_z+\hat{z}\tfrac{\partial}{\partial z}\times\vec{H}_t=\vec{J}_{et}+j\omega\varepsilon_t\vec{E}_t-\omega\varepsilon_g\hat{z}\times\vec{E}_t+j\omega\xi_t\vec{H}_t-\omega\xi_g\hat{z}\times\vec{H}_t\\&\nabla_t\times\hat{z}E_z+\hat{z}\tfrac{\partial}{\partial z}\times\vec{E}_t=-\vec{J}_{ht}-j\omega\mu_t\vec{H}_t+\omega\mu_g\hat{z}\times\vec{H}_t-j\omega\zeta_t\vec{E}_t+\omega\zeta_g\hat{z}\times\vec{E}_t\end{aligned}}^{Transverse\ Relations}$$

$$\overbrace{\begin{aligned}&\nabla\times\vec{H}_t=\hat{z}J_{ez}+\hat{z}j\omega\varepsilon_zE_z+\hat{z}j\omega\xi_zH_z\\&\nabla\times\vec{E}_t=-\hat{z}J_{hz}-j\omega\mu_zH_z-j\omega\zeta_zE_z\end{aligned}}^{Longitudinal\ Relations}\ \ or\ \begin{bmatrix}\hat{z}E_z\\ \hat{z}H_z\end{bmatrix}=\frac{1}{j\omega\underbrace{(\varepsilon_z\mu_z-\xi_z\zeta_z)}_{\Delta_z}}\begin{bmatrix}\mu_z & -\xi_z\\ -\zeta_z & \varepsilon_z\end{bmatrix}\begin{bmatrix}\nabla_t\times\vec{H}_t-\hat{z}J_{ez}\\ -\nabla_t\times\vec{E}_t-\hat{z}J_{hz}\end{bmatrix}$$

$$\vec{E}_t=\nabla_t\Phi+\nabla_t\times\hat{z}\theta\ ,\ \vec{H}_t=\nabla_t\pi+\nabla_t\times\hat{z}\psi$$
$$\vec{J}_{et}=\underbrace{\nabla_tu_e}_{lamellar}+\underbrace{\nabla_t\times\hat{z}v_e}_{rotational}\ ,\ \vec{J}_{ht}=\nabla_tu_h+\nabla_t\times\hat{z}v_h\ ,\ \nabla_t\perp\nabla_t\times\hat{z}\ \Rightarrow$$

Scalar Potential Based Analysis – Bianisotropic Gyrotropic Media

$$\begin{bmatrix} L_1 & L_2 \\ L_3 & L_4 \end{bmatrix}\begin{bmatrix} \psi \\ \theta \end{bmatrix}=\begin{bmatrix} s_1 \\ s_2 \end{bmatrix} \Rightarrow \begin{bmatrix} \psi \\ \theta \end{bmatrix}=\begin{bmatrix} L_1 & L_2 \\ L_3 & L_4 \end{bmatrix}^{-1}\begin{bmatrix} s_1 \\ s_2 \end{bmatrix} \quad [\underline{\overleftrightarrow{L}} \text{ is block } 1\times 1]$$

$$L_1=-\frac{\mu_z\Delta_t}{\mu_t\Delta_z}\nabla_t^2-\frac{\Delta_t}{\mu_t}\frac{\partial}{\partial z}(\frac{\mu_t}{\Delta_t}\frac{\partial}{\partial z})-\frac{\omega\Delta_t}{\mu_t}\frac{\partial}{\partial z}(\frac{\mu_t\xi_g-\mu_g\xi_t}{\Delta_t})+\omega(\frac{\mu_t\zeta_g-\mu_g\zeta_t}{\mu_t})\frac{\partial}{\partial z}-\omega^2[\Delta_t-\frac{\mu_g(\varepsilon_t\mu_g-\xi_g\zeta_t)+\zeta_g(\mu_t\xi_g-\mu_g\xi_t)}{\mu_t}]$$

$$L_2=-\frac{\xi_z\Delta_t}{\mu_t\Delta_z}\nabla_t^2-\frac{\Delta_t}{\mu_t}\frac{\partial}{\partial z}(\frac{\xi_t}{\Delta_t}\frac{\partial}{\partial z})-\frac{\omega\Delta_t}{\mu_t}\frac{\partial}{\partial z}(\frac{\varepsilon_g\mu_t-\xi_t\zeta_g}{\Delta_t})-\omega(\frac{\varepsilon_t\mu_g-\xi_t\zeta_g}{\mu_t})\frac{\partial}{\partial z}-\omega^2[\frac{\zeta_t\Delta_t}{\mu_t}-\frac{\mu_g(\varepsilon_t\zeta_g-\varepsilon_g\zeta_t)+\zeta_g(\varepsilon_g\mu_t-\xi_t\zeta_g)}{\mu_t}]$$

$$L_3=-\frac{\zeta_z\Delta_t}{\varepsilon_t\Delta_z}\nabla_t^2-\frac{\Delta_t}{\varepsilon_t}\frac{\partial}{\partial z}(\frac{\zeta_t}{\Delta_t}\frac{\partial}{\partial z})+\frac{\omega\Delta_t}{\varepsilon_t}\frac{\partial}{\partial z}(\frac{\varepsilon_t\mu_g-\xi_g\zeta_t}{\Delta_t})+\omega(\frac{\varepsilon_g\mu_t-\xi_g\zeta_t}{\varepsilon_t})\frac{\partial}{\partial z}-\omega^2[\frac{\xi_t\Delta_t}{\varepsilon_t}-\frac{\varepsilon_g(\mu_t\xi_g-\mu_g\xi_t)+\xi_g(\varepsilon_t\mu_g-\xi_g\zeta_t)}{\varepsilon_t}]$$

$$L_4=-\frac{\varepsilon_z\Delta_t}{\varepsilon_t\Delta_z}\nabla_t^2-\frac{\Delta_t}{\varepsilon_t}\frac{\partial}{\partial z}(\frac{\varepsilon_t}{\Delta_t}\frac{\partial}{\partial z})+\frac{\omega\Delta_t}{\varepsilon_t}\frac{\partial}{\partial z}(\frac{\varepsilon_t\zeta_g-\varepsilon_g\zeta_t}{\Delta_t})-\omega(\frac{\varepsilon_t\xi_g-\varepsilon_g\xi_t}{\varepsilon_t})\frac{\partial}{\partial z}-\omega^2[\Delta_t-\frac{\varepsilon_g(\varepsilon_g\mu_t-\xi_t\zeta_g)+\xi_g(\varepsilon_t\zeta_g-\varepsilon_g\zeta_t)}{\varepsilon_t}]$$

$$s_1=-\frac{\Delta_t}{\mu_t}\frac{\partial}{\partial z}(\frac{\mu_t}{\Delta_t}u_e)+\omega(\frac{\mu_t\zeta_g-\mu_g\zeta_t}{\mu_t})u_e+\frac{\mu_z\Delta_t}{\mu_t\Delta_z}J_{ez}+\frac{\Delta_t}{\mu_t}\frac{\partial}{\partial z}(\frac{\xi_t}{\Delta_t}u_h)+\omega(\frac{\varepsilon_t\mu_g-\xi_t\zeta_g}{\mu_t})u_h-\frac{j\omega\Delta_t}{\mu_t}v_h-\frac{\xi_z\Delta_t}{\mu_t\Delta_z}J_{hz}$$

$$s_2=-\frac{\Delta_t}{\varepsilon_t}\frac{\partial}{\partial z}(\frac{\zeta_t}{\Delta_t}u_e)+\omega(\frac{\varepsilon_g\mu_t-\xi_g\zeta_t}{\varepsilon_t})u_e-\frac{j\omega\Delta_t}{\varepsilon_t}v_e+\frac{\zeta_z\Delta_t}{\varepsilon_t\Delta_z}J_{ez}+\frac{\Delta_t}{\varepsilon_t}\frac{\partial}{\partial z}(\frac{\varepsilon_t}{\Delta_t}u_h)+\omega(\frac{\varepsilon_t\xi_g-\varepsilon_g\xi_t}{\varepsilon_t})u_h-\frac{\varepsilon_z\Delta_t}{\varepsilon_t\Delta_z}J_{hz}$$

$$\begin{bmatrix} \Phi \\ \pi \end{bmatrix}=\frac{1}{j\omega\Delta_t}\begin{bmatrix} \omega(\mu_t\xi_g-\mu_g\xi_t)\psi+\omega(\varepsilon_g\mu_t-\xi_t\zeta_g)\theta+\mu_t(\frac{\partial\psi}{\partial z}-u_e)+\xi_t(\frac{\partial\theta}{\partial z}+u_h) \\ \omega(\varepsilon_t\mu_g-\xi_g\zeta_t)\psi+\omega(\varepsilon_t\zeta_g-\varepsilon_g\zeta_t)\theta-\zeta_t(\frac{\partial\psi}{\partial z}-u_e)-\varepsilon_t(\frac{\partial\theta}{\partial z}+u_h) \end{bmatrix}$$

$$\begin{matrix} \vec{E}_t=\nabla_t\Phi+\nabla_t\times\hat{z}\theta \\ \vec{H}_t=\nabla_t\pi+\nabla_t\times\hat{z}\psi \end{matrix}, \quad \begin{matrix} \vec{E}_z=-\hat{z}\frac{\mu_z}{j\omega\Delta_z}\nabla_t^2\psi-\hat{z}\frac{\xi_z}{j\omega\Delta_z}\nabla_t^2\theta-\frac{\hat{z}\mu_z\hat{z}}{j\omega\Delta_z}\cdot\vec{J}_e+\frac{\hat{z}\xi_z\hat{z}}{j\omega\Delta_z}\cdot\vec{J}_h \\ \vec{H}_z=\hat{z}\frac{\varepsilon_z}{j\omega\Delta_z}\nabla_t^2\theta+\hat{z}\frac{\zeta_z}{j\omega\Delta_z}\nabla_t^2\psi-\frac{\hat{z}\varepsilon_z\hat{z}}{j\omega\Delta_z}\cdot\vec{J}_h+\frac{\hat{z}\zeta_z\hat{z}}{j\omega\Delta_z}\cdot\vec{J}_e \end{matrix}$$

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Field and Potential-Based Methods of Analysis – Key Take-Aways!

KEY Take-Aways

Constitutive relation form greatly influences analysis methodology!!

Scalar potentials should also be taught for SIMPLE media to aid in transition to complex media!!

Consider all factors before solving problems.

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Field and Potential-Based Methods of Analysis – Homework

Find alternative duality transformations.

Find the reciprocity relations for bianisotropic media in the C_{EB} formulation.

Under what conditions is image theory most useful for bianisotropic media?

$$\textit{Show the operator } \nabla\times\overleftrightarrow{I} = \overleftrightarrow{I}\times\nabla = (\hat{x}\hat{x}+\hat{y}\hat{y}+\hat{z}\hat{z})\times(\hat{x}\frac{\partial}{\partial x}+\hat{y}\frac{\partial}{\partial y}+\hat{z}\frac{\partial}{\partial z}) = \begin{bmatrix} 0 & -\frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & -\frac{\partial}{\partial x} \\ -\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{bmatrix}$$

Find the field analysis for anisotropic gyrotropic media having z-invariance.

Show how the scalar potential analysis simplifies for uniaxial anisotropic media.

$$\textit{Show } \vec{v}_1\cdot\overleftrightarrow{A}\cdot\vec{v}_2 = \vec{v}_2\cdot\overleftrightarrow{A}\cdot\vec{v}_1 \;\ldots \textit{if } \overleftrightarrow{A} = \overleftrightarrow{A}^T .$$

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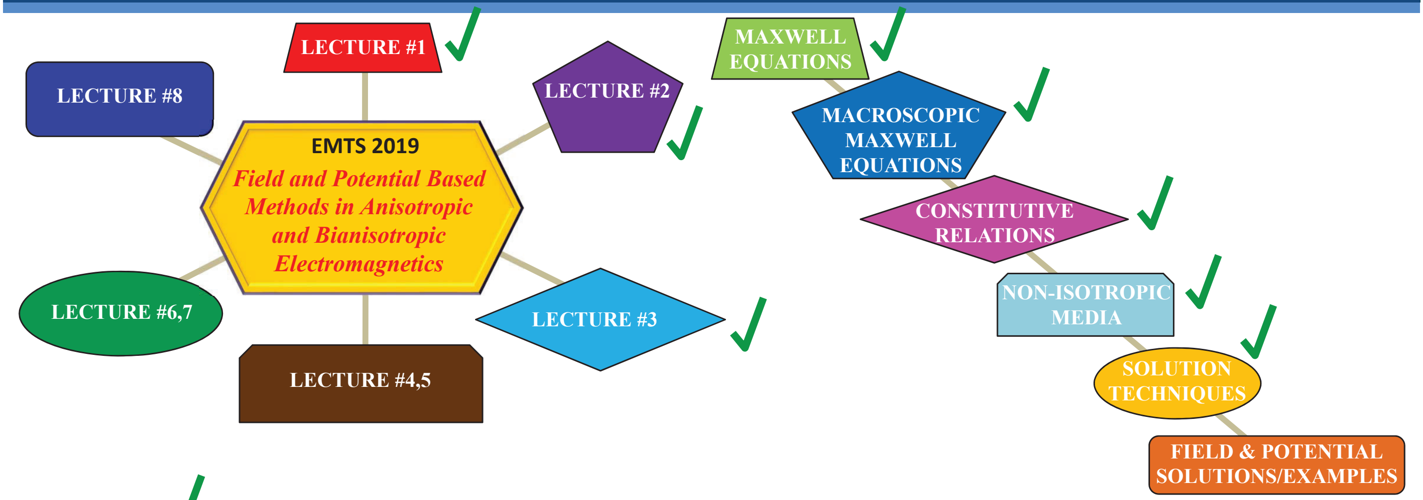
Overview – Lectures/Big Picture



LECTURE #1: ✓ Review various forms and regimes of validity of Maxwell's equations and constitutive relations.
LECTURE #2: ✓ Discuss factors that influence anisotropy and bianisotropy.
LECTURE #3: ✓ Discuss field and potential based methods for solving electromagnetic problems.
LECTURE #4,5: Provide several examples of field based solutions in complex media within and exterior to source regions.
LECTURE #6,7: Provide several examples of potential based solutions in complex media within and exterior to source regions.
LECTURE #8: Summary, conclusions and future research.

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2019 International Symposium on Electromagnetic Theory

LECTURE #4
Field-Based Examples – Source Free Region

Dr. Michael J. Havrilla
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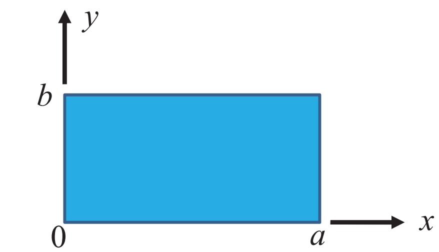
Rectangular Waveguide – Anisotropic Gyrotropic Media (y-invariance)



Rectangular Waveguide (PEC Walls, Source-free gyrotropic region)

$$\begin{bmatrix} \varepsilon_1 & 0 & j\varepsilon_3 \\ 0 & \varepsilon_2 & 0 \\ -j\varepsilon_3 & 0 & \varepsilon_1 \end{bmatrix} \qquad \begin{bmatrix} \mu_1 & 0 & j\mu_3 \\ 0 & \mu_2 & 0 \\ -j\mu_3 & 0 & \mu_1 \end{bmatrix}$$

$$TM^y \ (TE^x, TE^z)$$

$$\nabla_t^2 E_y + k^2 E_y = 0 \ , \ k^2 = \omega^2 \frac{(\mu_1^2 - \mu_3^2)\varepsilon_2}{\mu_1}$$

$$\begin{bmatrix} H_x \\ H_z \end{bmatrix} = \frac{1}{\omega^2(\mu_1^2 - \mu_3^2)} \begin{bmatrix} j\omega\mu_1 & \omega\mu_3 \\ -\omega\mu_3 & j\omega\varepsilon_1 \end{bmatrix} \begin{bmatrix} -\frac{\partial E_y}{\partial z} \\ \frac{\partial E_y}{\partial x} \end{bmatrix}$$

$$E_y(x,z) = \underbrace{f(x)g(z)}_{\substack{separation \\ of\ variables}} \rightarrow \frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial z^2} + k^2 E_y = 0 \Rightarrow E_y = (A \sin k_x x + B \cos k_x x)(Ce^{-jk_z z} + De^{jk_z z})$$

$$E_y(0,z) = 0 \ \forall z \Rightarrow B = 0$$

$$E_y(a,z) = 0 \ \forall z \Rightarrow k_x = k_{xm} = \frac{m\pi}{a} \cdots \begin{matrix} m = 1,2,3,\ldots,\infty \\ (m = 0 \ trivial) \end{matrix} \Rightarrow E_{ym}(x,z) = \sin k_{xm} x (\overbrace{AC}^{A_m^+} e^{-jk_{zm} z} + \overbrace{AD}^{A_m^-} e^{jk_{zm} z})$$

$$k_{zm} = \sqrt{k^2 - k_{xm}^2}$$

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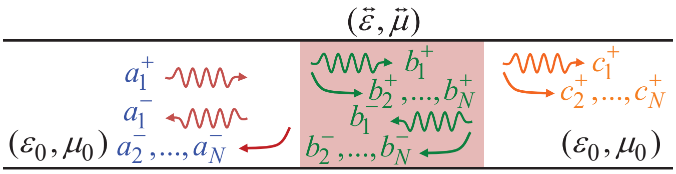
Rectangular Waveguide – Anisotropic Gyrotropic Media (y-invariance)

$$E_{ym} = \sin k_{xm}x(A_m^+ e^{-jk_{zm}z} + A_m^- e^{jk_{zm}z}) \rightarrow \begin{bmatrix} H_x \\ H_z \end{bmatrix} = \frac{1}{\omega^2(\mu_1^2-\mu_3^2)} \begin{bmatrix} j\omega\mu_1 & \omega\mu_3 \\ -\omega\mu_3 & j\omega\varepsilon_1 \end{bmatrix} \begin{bmatrix} -\frac{\partial E_y}{\partial z} \\ \frac{\partial E_y}{\partial x} \end{bmatrix} \Rightarrow$$

$$H_{xm} = \frac{1}{\omega^2(\mu_1^2-\mu_3^2)}[-\omega\mu_1 k_{zm}\sin k_{xm}x(A_m^+ e^{-jk_{zm}z} - A_m^- e^{jk_{zm}z}) + \omega\mu_3 k_{xm}\cos k_{xm}x(A_m^+ e^{-jk_{zm}z} + A_m^- e^{jk_{zm}z})]$$

$$H_{zm} = \frac{1}{\omega^2(\mu_1^2-\mu_3^2)}[-j\omega\mu_3 k_{zm}\sin k_{xm}x(A_m^+ e^{-jk_{zm}z} - A_m^- e^{jk_{zm}z}) + j\omega\varepsilon_1 k_{xm}\cos k_{xm}x(A_m^+ e^{-jk_{zm}z} + A_m^- e^{jk_{zm}z})]$$



Rectangular waveguide filled with a gyrotropic sample.
(Can solve using the mode-matching technique)

J. Tang, et al., "Characterization of Y-Bias Ferrite Materials for Tunable Antenna Applications Using a Partially-Filled Rectangular Waveguide," Transactions on Antennas and Propagation, vol. 65, no. 10, pp. 5279-5288, October 2017

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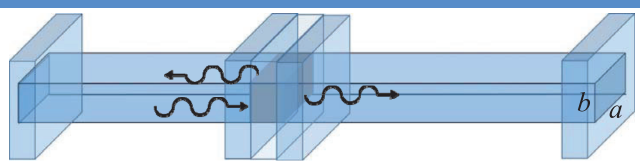
Rectangular Waveguide – Anisotropic Biaxial Media (y-invariance)



TE_{10} $Mode: \vec{E} = \hat{y}E_0\sin(\pi x/a)$
$\ldots a\ y-invariant\ mode$

$$\begin{matrix} \vec{\varepsilon} = \hat{x}\hat{x}\varepsilon_{xx} + \hat{y}\hat{y}\varepsilon_{yy} + \hat{z}\hat{z}\varepsilon_{zz} \\ \vec{\mu} = \hat{x}\hat{x}\mu_{xx} + \hat{y}\hat{y}\mu_{yy} + \hat{z}\hat{z}\mu_{zz} \end{matrix}, \begin{matrix} \vec{\xi} = 0 \\ \vec{\zeta} = 0 \end{matrix}, \begin{matrix} \vec{J}_e = 0 \\ \vec{J}_h = 0 \end{matrix} \quad y-invariant \Rightarrow$$

TE^z Modes

$$\frac{\mu_x}{\mu_z}\frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial z^2} + \omega^2\varepsilon_y\mu_x E_y = 0$$

$$H_x = \frac{1}{j\omega\mu_x}\frac{\partial E_y}{\partial z}, \quad H_z = -\frac{1}{j\omega\mu_z}\frac{\partial E_y}{\partial x}$$

$$\left.\begin{matrix} E_{ym} = \sin k_x x(A_m^+ e^{-jk_{zm}z} + A_m^- e^{jk_{zm}z}) \\ H_{xm} = -\frac{k_{zm}}{\omega\mu_x}\sin k_{xm}x(A_m^+ e^{-jk_{zm}z} - A_m^- e^{jk_{zm}z}) \\ H_{zm} = -\frac{k_{xm}}{j\omega\mu_z}\cos k_{xm}x(A_m^+ e^{-jk_{zm}z} + A_m^- e^{jk_{zm}z}) \end{matrix}\right\}\ldots \begin{matrix} using\ separation\ of\ variables \\ k_{zm} = \sqrt{\omega^2\varepsilon_y\mu_x - \frac{\mu_x}{\mu_z}k_{xm}^2} \\ k_{xm} = \frac{m\pi}{a}\ldots m = 1,2,3,\ldots \end{matrix}$$

TM^z Modes

$$\frac{\varepsilon_x}{\varepsilon_z}\frac{\partial^2 H_y}{\partial x^2} + \frac{\partial^2 H_y}{\partial z^2} + \omega^2\varepsilon_x\mu_y H_y = 0$$

$$E_x = -\frac{1}{j\omega\varepsilon_x}\frac{\partial H_y}{\partial z}, \quad E_z = \frac{1}{j\omega\varepsilon_z}\frac{\partial H_y}{\partial x}$$

(boundary condition enforcement leads to a zero field)

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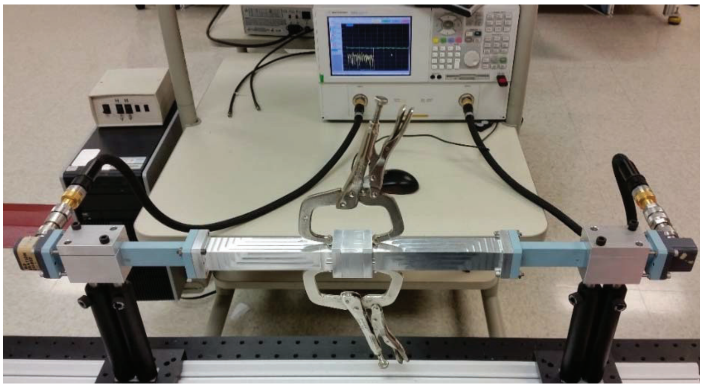
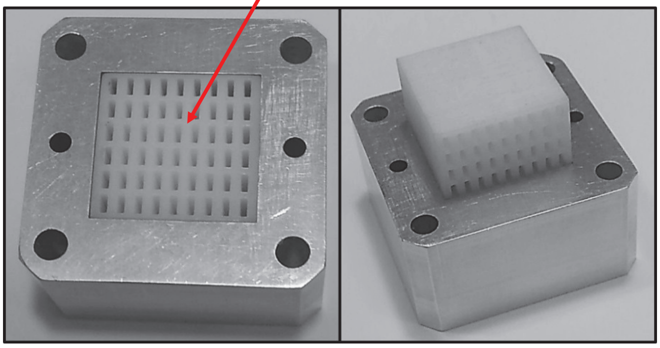
Rectangular Waveguide – Anisotropic Biaxial Media (y-invariance) - Application

Measurement of biaxial media*.

3D printed sample with orthorhombic symmetry infused into the design!





*A. Knisely, "Biaxial Anisotropic Material Characterization using Rectangular to Square Waveguide", AMTA, 2014.

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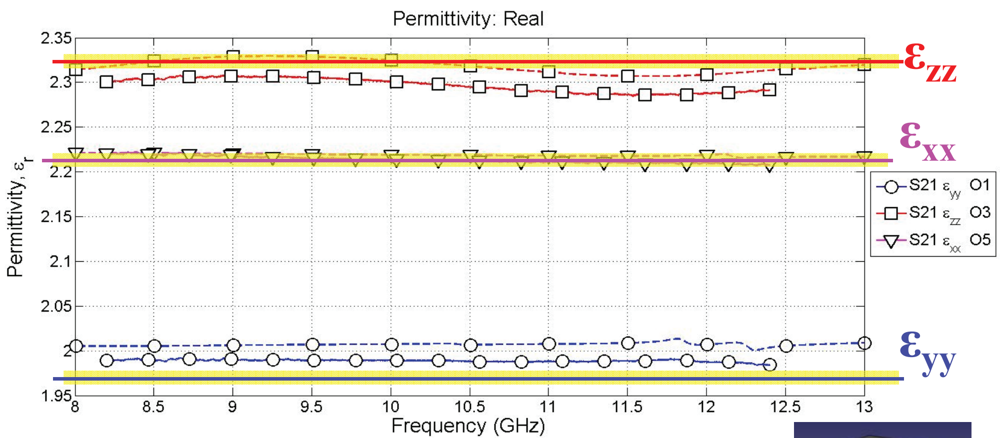
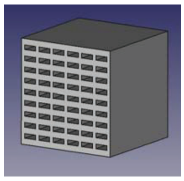
Rectangular Waveguide – Anisotropic Biaxial Media (y-invariance) - Application

Measured Sample Permittivity

Comparison: Test Data, CST and Lumped Circuit Equiv.





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Plane Waves – Bianisotropic Media

$$\overleftrightarrow{W}_E \cdot \vec{E} = [(\nabla \times \overleftrightarrow{I} - j\omega\overleftrightarrow{\xi}) \cdot \overleftrightarrow{\mu}^{-1} \cdot (\nabla \times \overleftrightarrow{I} + j\omega\overleftrightarrow{\zeta}) - \omega^2 \overleftrightarrow{\varepsilon}] \cdot \vec{E} = -j\omega\vec{J}_e - (\nabla \times \overleftrightarrow{I} - j\omega\overleftrightarrow{\xi}) \cdot \overleftrightarrow{\mu}^{-1} \cdot \vec{J}_h$$

$$\vec{H} = -\frac{\overleftrightarrow{\mu}^{-1} \cdot (\nabla \times \overleftrightarrow{I} + j\omega\overleftrightarrow{\zeta}) \cdot \vec{E}}{j\omega} - \frac{\overleftrightarrow{\mu}^{-1} \cdot \vec{J}_h}{j\omega} \qquad \vec{J}_e, \vec{J}_h = 0 \;\; and \;\; \begin{matrix} \vec{E} = \vec{E}_0 e^{-j\vec{k}\cdot\vec{r}} \\ \vec{H} = \vec{H}_0 e^{-j\vec{k}\cdot\vec{r}} \end{matrix} \Rightarrow \nabla \to -j\vec{k} \;\; ...thus$$

$$\boxed{\begin{matrix} \overleftrightarrow{W}_E \cdot \vec{E} = 0 \;\; or \;\; \overleftrightarrow{W}_E \cdot \vec{E}_0 = 0 \\ \vec{H} = \frac{1}{\omega}\overleftrightarrow{\mu}^{-1} \cdot (\overleftrightarrow{k} - \omega\overleftrightarrow{\zeta}) \cdot \vec{E} \\ \vec{H}_0 = \frac{1}{\omega}\overleftrightarrow{\mu}^{-1} \cdot (\overleftrightarrow{k} - \omega\overleftrightarrow{\zeta}) \cdot \vec{E}_0 \end{matrix} \quad , \quad \begin{matrix} \overleftrightarrow{W}_E = (\overleftrightarrow{k} + \omega\overleftrightarrow{\xi}) \cdot \overleftrightarrow{\mu}^{-1} \cdot (\overleftrightarrow{k} - \omega\overleftrightarrow{\zeta}) + \omega^2 \overleftrightarrow{\varepsilon} \\ \overleftrightarrow{I} = \hat{x}\hat{x} + \hat{y}\hat{y} + \hat{z}\hat{z} \;\; , \;\; \overleftrightarrow{k} = \vec{k} \times \overleftrightarrow{I} \\ [known\ as\ the\ kEH\ formulation] \end{matrix}}$$

$$\det \overleftrightarrow{W}_E = 0 \;\; ...determines\ allowed\ eigenvalues\ (i.e.,\ propagation\ constants\ \vec{k}_{eigen})$$

$$\overleftrightarrow{W}_E \cdot \vec{E}_0 \Big|_{\vec{k}_{eigen}} = 0 \;\; ...determines\ eigenvectors\ (i.e.,\ polarization\ states\ \vec{E}_{0,eigen})$$

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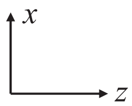
Plane Waves – Anisotropic Biaxial Media (Normal Incidence) Example



$$\begin{matrix} \overleftrightarrow{\varepsilon} = \hat{x}\hat{x}\varepsilon_{xx} + \hat{y}\hat{y}\varepsilon_{yy} + \hat{z}\hat{z}\varepsilon_{zz} \\ \overleftrightarrow{\mu} = \hat{x}\hat{x}\mu_{xx} + \hat{y}\hat{y}\mu_{yy} + \hat{z}\hat{z}\mu_{zz} \end{matrix} \;\; , \;\; \begin{matrix} \overleftrightarrow{\xi} = 0 \\ \overleftrightarrow{\zeta} = 0 \end{matrix} \;\; and \;\; \vec{k} = \hat{z}k_z \;\; \Rightarrow$$

$$\overleftrightarrow{W}_E \cdot \vec{E}_0 = \begin{bmatrix} \omega^2\varepsilon_{xx} - (k_z^2/\mu_{yy}) & 0 & 0 \\ 0 & \omega^2\varepsilon_{yy} - (k_z^2/\mu_{xx}) & 0 \\ 0 & 0 & \omega^2\varepsilon_{zz} \end{bmatrix} \begin{bmatrix} E_{0x} \\ E_{0y} \\ E_{0z} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \;\; , \;\; \vec{H}_0 = \begin{bmatrix} 0 & -\frac{k_z}{\omega\mu_{xx}} & 0 \\ \frac{k_z}{\omega\mu_{yy}} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} E_{0x} \\ E_{0y} \\ E_{0z} \end{bmatrix}$$

$$\det \overleftrightarrow{W}_E = (\omega^2\varepsilon_{xx} - \frac{k_z^2}{\mu_{yy}})(\omega^2\varepsilon_{yy} - \frac{k_z^2}{\mu_{xx}})\omega^2\varepsilon_{zz} = 0 \;\; \Rightarrow \;\; \underbrace{\begin{matrix} k_z = k_z^{\|\pm} = \pm\omega\sqrt{\varepsilon_{xx}\mu_{yy}} = \pm k_z^{\|} \\ k_z = k_z^{\perp\pm} = \pm\omega\sqrt{\varepsilon_{yy}\mu_{xx}} = \pm k_z^{\perp} \end{matrix}}_{\substack{physical\ insight\ implies\ these\ \perp,\|\ (to\ x-z\ plane) \\ polarizations\ are\ expected\ (can\ you\ explain)?}}$$

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Plane Waves – Anisotropic Biaxial Media

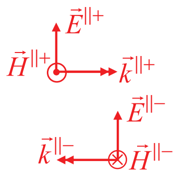
$$\vec{W}_E\cdot\vec{E}_0\Big|_{k_z=+k_z^{\|}=\omega\sqrt{\varepsilon_{xx}\mu_{yy}}}=0 \Rightarrow \begin{bmatrix} \omega^2\varepsilon_{xx}-(\omega^2\varepsilon_{xx}\mu_{yy}/\mu_{yy}) & 0 & 0 \\ 0 & \omega^2\varepsilon_{yy}-(\omega^2\varepsilon_{xx}\mu_{yy}/\mu_{xx}) & 0 \\ 0 & 0 & \omega^2\varepsilon_{zz} \end{bmatrix}\begin{bmatrix} E_{0x}^{\|+} \\ E_{0y}^{\|+} \\ E_{0z}^{\|+} \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \vec{E}^{\|+}=\vec{E}_0^{\|+}e^{-j\vec{k}^{\|+}\cdot\vec{r}}=(\hat{x}E_{0x}^{\|+}+\hat{y}\overbrace{E_{0y}^{\|+}}^{0}+\hat{z}\overbrace{E_{0z}^{\|+}}^{0})e^{-j(\hat{x}\overbrace{k_x^{\|+}}^{0}+\hat{y}\overbrace{k_y^{\|+}}^{0}+\hat{z}\overbrace{k_z^{\|+}}^{+k_z^{\|}})\cdot(\hat{x}x+\hat{y}y+\hat{z}z)} \Rightarrow \boxed{\vec{E}^{\|+}=\hat{x}E_{0x}^{\|+}e^{-jk_z^{\|}z}}$$

$$\Rightarrow \vec{H}^{\|+}=\frac{1}{\omega}\overleftrightarrow{\mu}^{-1}\cdot(\vec{k}^{\|+}\times\overleftrightarrow{I}-\omega\overbrace{\overleftrightarrow{\zeta}}^{0})\cdot\vec{E}^{\|+}=\frac{\hat{z}\times\vec{E}^{\|+}}{Z^{\|}}\ \therefore\ \boxed{\vec{H}^{\|+}=\frac{\hat{z}\times\vec{E}^{\|+}}{Z^{\|}}\ ,\ Z^{\|}=\frac{\omega\mu_{yy}}{k_z^{\|}}=\sqrt{\frac{\mu_{yy}}{\varepsilon_{xx}}}}$$

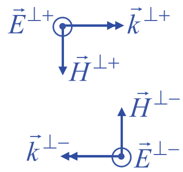
$$\vec{E}^{\|\pm}=\hat{x}E_{0x}^{\|\pm}e^{\mp jk_z^{\|}z}$$

$$\vec{H}^{\|\pm}=\pm\frac{\hat{z}\times\vec{E}^{\|\pm}}{Z^{\|}}\ ,\ Z^{\|}=\sqrt{\frac{\mu_{yy}}{\varepsilon_{xx}}}$$



$$\vec{E}^{\perp\pm}=\hat{y}E_{0y}^{\perp\pm}e^{\mp jk_z^{\perp}z}$$

$$\vec{H}^{\perp\pm}=\pm\frac{\hat{z}\times\vec{E}^{\perp\pm}}{Z^{\perp}}\ ,\ Z^{\perp}=\sqrt{\frac{\mu_{xx}}{\varepsilon_{yy}}}$$



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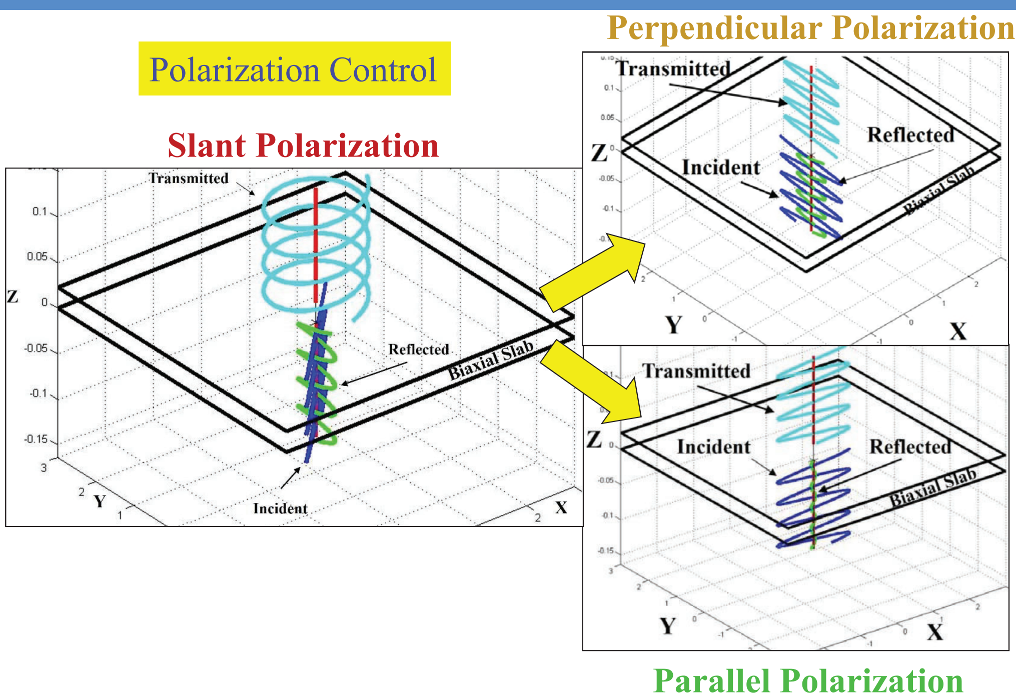
Plane Waves – Anisotropic Biaxial Media - Application



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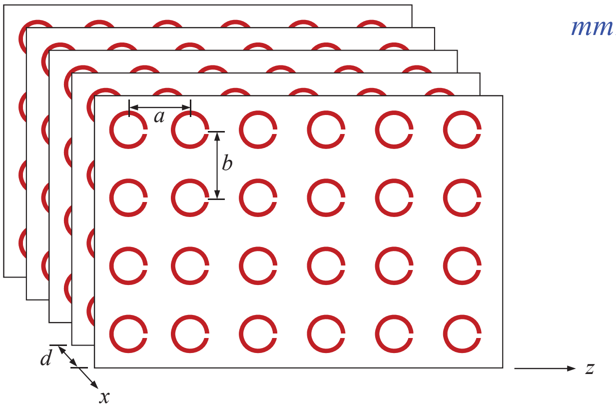
Plane Waves – Bianisotropic Media

$\vec{k} = \hat{z}k_z$

normal incidence

$\bar{\bar{k}} = \vec{k} \times \bar{\bar{I}} = \hat{y}\hat{x}k_z - \hat{x}\hat{y}k_z$



$mm21' \; (1, m_x, m_y, 2_z, 1', m'_x, m'_y, 2'_z)$

$$\begin{bmatrix} \varepsilon_x & 0 & 0 \\ 0 & \varepsilon_y & 0 \\ 0 & 0 & \varepsilon_z \end{bmatrix}, \begin{bmatrix} \mu_x & 0 & 0 \\ 0 & \mu_y & 0 \\ 0 & 0 & \mu_z \end{bmatrix}$$

$$\begin{bmatrix} 0 & \xi_{xy} & 0 \\ \xi_{yx} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \bar{\bar{\zeta}} = -\bar{\bar{\xi}}^T$$

$$\bar{\bar{W}}_E = (\bar{\bar{k}} + \omega\bar{\bar{\xi}}) \cdot \bar{\bar{\mu}}^{-1} \cdot (\bar{\bar{k}} - \omega\bar{\bar{\zeta}}) + \omega^2 \bar{\bar{\varepsilon}} = \begin{bmatrix} -\frac{1}{\mu_y}(k_z^2 - \omega^2\xi_{xy}^2) + \omega^2\varepsilon_x & 0 & 0 \\ 0 & -\frac{1}{\mu_x}(k_z^2 - \omega^2\xi_{yx}^2) + \omega^2\varepsilon_y & 0 \\ 0 & 0 & \omega^2\varepsilon_z \end{bmatrix}$$

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Plane Waves – Bianisotropic Media

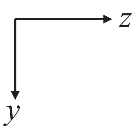
$$\bar{\bar{W}}_E \cdot \vec{E}_0 = \begin{bmatrix} -\frac{1}{\mu_y}(k_z^2 - \omega^2\xi_{xy}^2) + \omega^2\varepsilon_x & 0 & 0 \\ 0 & -\frac{1}{\mu_x}(k_z^2 - \omega^2\xi_{yx}^2) + \omega^2\varepsilon_y & 0 \\ 0 & 0 & \omega^2\varepsilon_z \end{bmatrix} \begin{bmatrix} E_{0x} \\ E_{0y} \\ E_{0z} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\det \bar{\bar{W}}_E = 0 \;\Rightarrow\; \begin{matrix} k_{z1}^{\pm} = \pm k_{z1} = \pm\omega\sqrt{\varepsilon_y\mu_x + \xi_{yx}^2} \\ k_{z2}^{\pm} = \pm k_{z2} = \pm\omega\sqrt{\varepsilon_x\mu_y + \xi_{xy}^2} \end{matrix}$$

$$k_z = \pm k_{z1} \;\rightarrow\; \bar{\bar{W}}_E \cdot \vec{E}_0 = 0 \;\Rightarrow$$

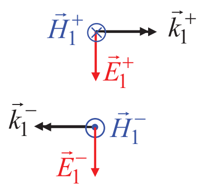
$$\vec{E}_1^{\pm} = \vec{E}_{01}^{\pm} e^{-j\vec{k}_1^{\pm}\cdot\vec{r}} = \hat{y}E_{01}^{\pm} e^{\mp jk_{z1}z} \; ...\underbrace{lin.\ pol.}_{expected\,?}$$

$$\vec{E}_1^{\pm}, k_z = \pm k_{z1} \;\rightarrow\; \vec{H} = \frac{1}{\omega}\bar{\bar{\mu}}^{-1} \cdot (\vec{k} \times \bar{\bar{I}} - \omega\bar{\bar{\zeta}}) \cdot \vec{E} \;\Rightarrow$$



$$\underbrace{\vec{H}_1^{\pm} = \mp\hat{x}\frac{E_{01}^{\pm}e^{\mp jk_{z1}z}}{Z_1^{\pm}} = \frac{\hat{k}_1^{\pm} \times \vec{E}_1^{\pm}}{Z_1^{\pm}}}_{consistent\ with\ Poynting\ vector}, \quad \hat{k}_1^{\pm} = \pm\hat{z}, \quad Z_1^{\pm} = \frac{\omega\mu_x}{k_{z1} \mp \omega\xi_{yx}}$$

Interesting!



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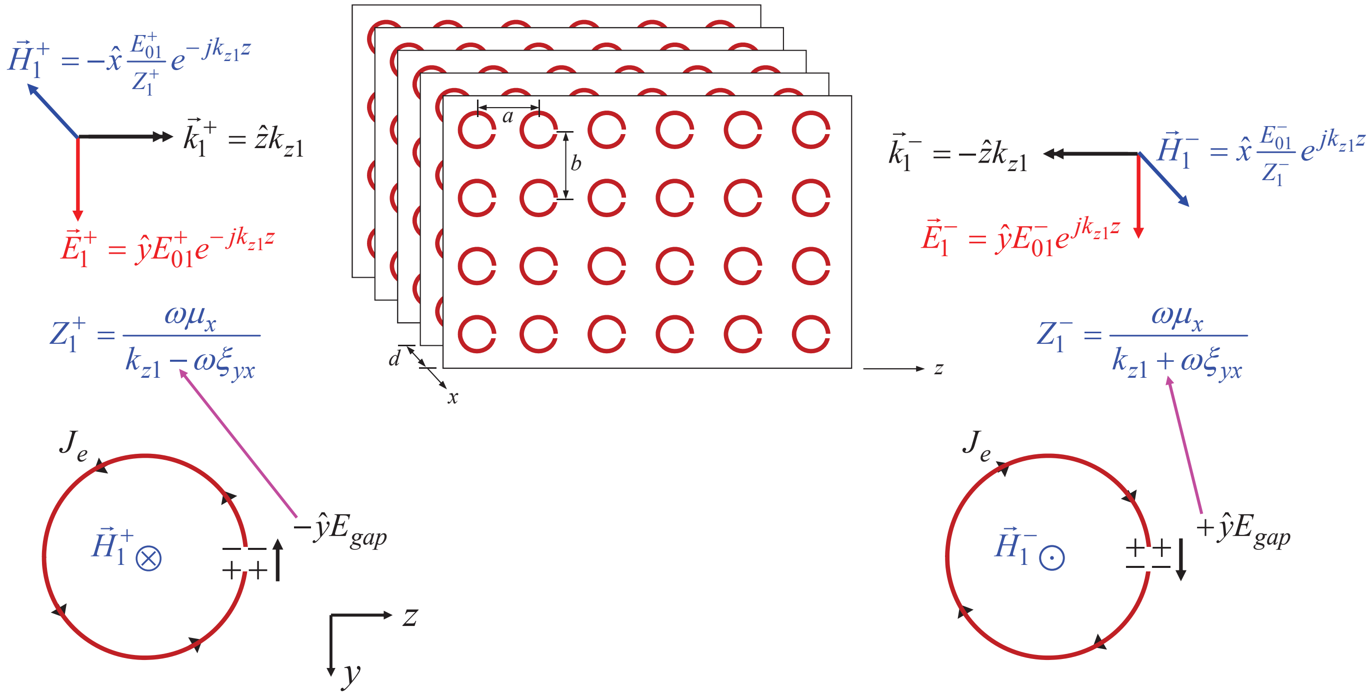
Plane Waves – Bianisotropic Media



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Plane Waves – kDB System (kEH Review)

$$\vec{E} = \vec{E}_0 e^{-j\vec{k}\cdot\vec{r}}, \quad \vec{H} = \vec{H}_0 e^{-j\vec{k}\cdot\vec{r}}, \quad \vec{D} = \vec{D}_0 e^{-j\vec{k}\cdot\vec{r}}, \quad \vec{B} = \vec{B}_0 e^{-j\vec{k}\cdot\vec{r}}$$

$$\rightarrow \left. \begin{array}{c} \nabla\times\vec{E} = -\vec{J}_h - j\omega\vec{B} \\ \nabla\times\vec{H} = \vec{J}_e + j\omega\vec{D} \\ \nabla\cdot\vec{D} = \rho_e \\ \nabla\cdot\vec{B} = \rho_h \end{array} \right|_{\substack{\vec{J}_e,\vec{J}_h,\rho_e,\rho_h=0 \\ \nabla\rightarrow -j\vec{k}}} \Rightarrow \boxed{\begin{array}{c} \vec{k}\times\vec{E} = \omega\vec{B} \\ \vec{k}\times\vec{H} = -\omega\vec{D} \\ \vec{k}\cdot\vec{D} = 0 \\ \vec{k}\cdot\vec{B} = 0 \end{array}}$$

Not so fun!

$$\begin{array}{c} \vec{k}\times\vec{E} = \omega\vec{B} \\ \vec{k}\times\vec{H} = -\omega\vec{D} \end{array}, \quad \begin{array}{c} \vec{D} = \bar{\varepsilon}\cdot\vec{E} + \bar{\xi}\cdot\vec{H} \\ \vec{B} = \bar{\zeta}\cdot\vec{E} + \bar{\mu}\cdot\vec{H} \end{array} \Rightarrow \overbrace{[(\vec{k}+\omega\bar{\xi})\cdot\bar{\mu}^{-1}\cdot(\vec{k}-\omega\bar{\zeta}) + \omega^2\bar{\varepsilon}]}^{\bar{W}_E \text{ (3×3 matrix!!!)}}\cdot\overbrace{\vec{E}}^{\vec{E}_0 e^{-j\vec{k}\cdot\vec{r}}} = 0 \ldots$$

$$\vec{H} = \frac{1}{\omega}\bar{\mu}^{-1}\cdot(\vec{k}\times\bar{I} - \omega\bar{\zeta})\cdot\vec{E}$$

kEH formulation (as already discussed)

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Plane Waves – kDB System

$$\vec{k}\times\vec{E}=\omega\vec{B}$$
$$\vec{k}\times\vec{H}=-\omega\vec{D}\qquad \vec{k}\cdot\vec{D}=0\;\Rightarrow\; D_k=0\;\Rightarrow\;\vec{D}=\vec{D}_t+\hat{k}D_k=\vec{D}_t$$
$$\vec{k}\cdot\vec{D}=0\qquad \vec{k}\cdot\vec{B}=0\;\Rightarrow\; B_k=0\;\Rightarrow\;\vec{B}=\vec{B}_t+\hat{k}B_k=\vec{B}_t$$
$$\vec{k}\cdot\vec{B}=0$$

$$\begin{aligned}\overbrace{\vec{k}}^{\hat{k}k}\times\overbrace{\vec{E}}^{\vec{E}_t+\hat{k}E_k}&=\omega\overbrace{\vec{B}}^{\vec{B}_t+\hat{k}B_k}\\ \underbrace{\vec{k}}_{\hat{k}k}\times\underbrace{\vec{H}}_{\vec{H}_t+\hat{k}H_k}&=-\omega\underbrace{\vec{D}}_{\vec{D}_t+\hat{k}D_k}\end{aligned}\;\Rightarrow\;\begin{aligned}\vec{k}\times\vec{E}_t&=\omega\vec{B}_t\\ \vec{k}\times\vec{H}_t&=-\omega\vec{D}_t\end{aligned}$$

$$\overleftrightarrow{M}=\overleftrightarrow{M}_{tt}+\overleftrightarrow{M}_{tk}+\overleftrightarrow{M}_{kt}+\overleftrightarrow{M}_{kk}=\begin{bmatrix}\overbrace{\overleftrightarrow{M}_{tt}}^{2\times2} & \overbrace{\overleftrightarrow{M}_{tk}}^{2\times1}\\ \underbrace{\overleftrightarrow{M}_{kt}}_{1\times2} & \underbrace{\overleftrightarrow{M}_{kk}}_{1\times1}\end{bmatrix}\ldots\begin{array}{c}\textit{general } 3\times3 \textit{ matrix decomposition}\\ (t,k=\textit{transverse to, along } \hat{k} \textit{ direction}).\end{array}$$

$$\begin{aligned}\overbrace{\vec{E}}^{\vec{E}_t+\hat{k}E_k}&=\overbrace{\overleftrightarrow{\kappa}}^{\overleftrightarrow{\kappa}_{tt}+\overleftrightarrow{\kappa}_{tk}+\overleftrightarrow{\kappa}_{kt}+\overleftrightarrow{\kappa}_{kk}}\cdot\overbrace{\vec{D}}^{\vec{D}_t+\overbrace{\hat{k}D_k}^{=0}}+\overbrace{\overleftrightarrow{\chi}}^{\overleftrightarrow{\chi}_{tt}+\overleftrightarrow{\chi}_{tk}+\overleftrightarrow{\chi}_{kt}+\overleftrightarrow{\chi}_{kk}}\cdot\overbrace{\vec{B}}^{\vec{B}_t+\overbrace{\hat{k}B_k}^{=0}}\\ \underbrace{\vec{H}}_{\vec{H}_t+\hat{k}H_k}&=\underbrace{\overleftrightarrow{\gamma}}_{\overleftrightarrow{\gamma}_{tt}+\overleftrightarrow{\gamma}_{tk}+\overleftrightarrow{\gamma}_{kt}+\overleftrightarrow{\gamma}_{kk}}\cdot\underbrace{\vec{D}}_{\vec{D}_t+\overbrace{\hat{k}D_k}^{=0}}+\underbrace{\overleftrightarrow{\nu}}_{\overleftrightarrow{\nu}_{tt}+\overleftrightarrow{\nu}_{tk}+\overleftrightarrow{\nu}_{kt}+\overleftrightarrow{\nu}_{kk}}\cdot\underbrace{\vec{B}}_{\vec{B}_t+\overbrace{\hat{k}B_k}^{=0}}\end{aligned}\;\Rightarrow\;\begin{aligned}\vec{E}_t&=\overleftrightarrow{\kappa}_{tt}\cdot\vec{D}_t+\overleftrightarrow{\chi}_{tt}\cdot\vec{B}_t\\ \vec{E}_k&=\overleftrightarrow{\kappa}_{kt}\cdot\vec{D}_t+\overleftrightarrow{\chi}_{kt}\cdot\vec{B}_t\\ \vec{H}_t&=\overleftrightarrow{\gamma}_{tt}\cdot\vec{D}_t+\overleftrightarrow{\nu}_{tt}\cdot\vec{B}_t\\ \vec{H}_k&=\overleftrightarrow{\gamma}_{kt}\cdot\vec{D}_t+\overleftrightarrow{\nu}_{kt}\cdot\vec{B}_t\end{aligned}$$

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Plane Waves – kDB System

$$\begin{aligned}\vec{E}_t&=\overleftrightarrow{\kappa}_{tt}\cdot\vec{D}_t+\overleftrightarrow{\chi}_{tt}\cdot\vec{B}_t\\ \vec{H}_t&=\overleftrightarrow{\gamma}_{tt}\cdot\vec{D}_t+\overleftrightarrow{\nu}_{tt}\cdot\vec{B}_t\end{aligned}\;\rightarrow\;\begin{aligned}\vec{k}\times\vec{E}_t&=\omega\vec{B}_t\\ \vec{k}\times\vec{H}_t&=-\omega\vec{D}_t\end{aligned}\;\Rightarrow\;\begin{aligned}\vec{k}\times(\overleftrightarrow{\kappa}_{tt}\cdot\vec{D}_t+\overleftrightarrow{\chi}_{tt}\cdot\vec{B}_t)&=\omega\vec{B}_t=\omega\overleftrightarrow{I}_t\cdot\vec{B}_t\ (1)\\ \vec{k}\times(\overleftrightarrow{\gamma}_{tt}\cdot\vec{D}_t+\overleftrightarrow{\nu}_{tt}\cdot\vec{B}_t)&=-\omega\vec{D}_t=-\omega\overleftrightarrow{I}_t\cdot\vec{D}_t\ (2)\end{aligned}$$

$$(1)\;\Rightarrow\;(\vec{k}\times\overleftrightarrow{\kappa}_{tt})\cdot\vec{D}_t=(\omega\overleftrightarrow{I}_t-\vec{k}\times\overleftrightarrow{\chi}_{tt})\cdot\vec{B}_t\;\Rightarrow\;\vec{B}_t=(\omega\overleftrightarrow{I}_t-\vec{k}\times\overleftrightarrow{\chi}_{tt})^{-1}\cdot(\vec{k}\times\overleftrightarrow{\kappa}_{tt})\cdot\vec{D}_t\ \ (3)$$

$$(3)\;\rightarrow\;(2)\;\Rightarrow\;(\vec{k}\times\overleftrightarrow{\gamma}_{tt})\cdot\vec{D}_t+(\vec{k}\times\overleftrightarrow{\nu}_{tt})\cdot(\omega\overleftrightarrow{I}_t-\vec{k}\times\overleftrightarrow{\chi}_{tt})^{-1}\cdot(\vec{k}\times\overleftrightarrow{\kappa}_{tt})\cdot\vec{D}_t=-\omega\overleftrightarrow{I}_t\cdot\vec{D}_t\;\Rightarrow$$
$$[(\omega\overleftrightarrow{I}_t+\vec{k}\times\overleftrightarrow{\gamma}_{tt})+(\vec{k}\times\overleftrightarrow{\nu}_{tt})\cdot(\omega\overleftrightarrow{I}_t-\vec{k}\times\overleftrightarrow{\chi}_{tt})^{-1}\cdot(\vec{k}\times\overleftrightarrow{\kappa}_{tt})]\cdot\vec{D}_t=0$$

$$\left.\begin{aligned}\underbrace{\overleftrightarrow{W}_{Dt}}_{2\times2!!!}\cdot\vec{D}_t&=[(\omega\overleftrightarrow{I}_t+\vec{k}\times\overleftrightarrow{\gamma}_{tt})+(\vec{k}\times\overleftrightarrow{\nu}_{tt})\cdot(\omega\overleftrightarrow{I}_t-\vec{k}\times\overleftrightarrow{\chi}_{tt})^{-1}\cdot(\vec{k}\times\overleftrightarrow{\kappa}_{tt})]\cdot\vec{D}_t=0\\ \vec{B}_t&=(\omega\overleftrightarrow{I}_t-\vec{k}\times\overleftrightarrow{\chi}_{tt})^{-1}\cdot(\vec{k}\times\overleftrightarrow{\kappa}_{tt})\cdot\vec{D}_t\quad,\quad(\vec{D}_t\ \textit{defines polarization!})\end{aligned}\right\}\ldots\ \textit{kDB formulation}$$

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Field Based Examples (Source-Free Region) – Key Take-Aways!

KEY Take-Aways

Consider all factors (symmetry, invariance, etc.) before solving problems.

kDB system can offer mathematical simplification for complex media.

Take time to make sure results make physical sense!

Field Based Examples (Source-Free Region) – Homework

For the case of a rectangular waveguide filled with anisotropic gyrotropic media and assuming y-invariance, show the TE^y modes are zero.

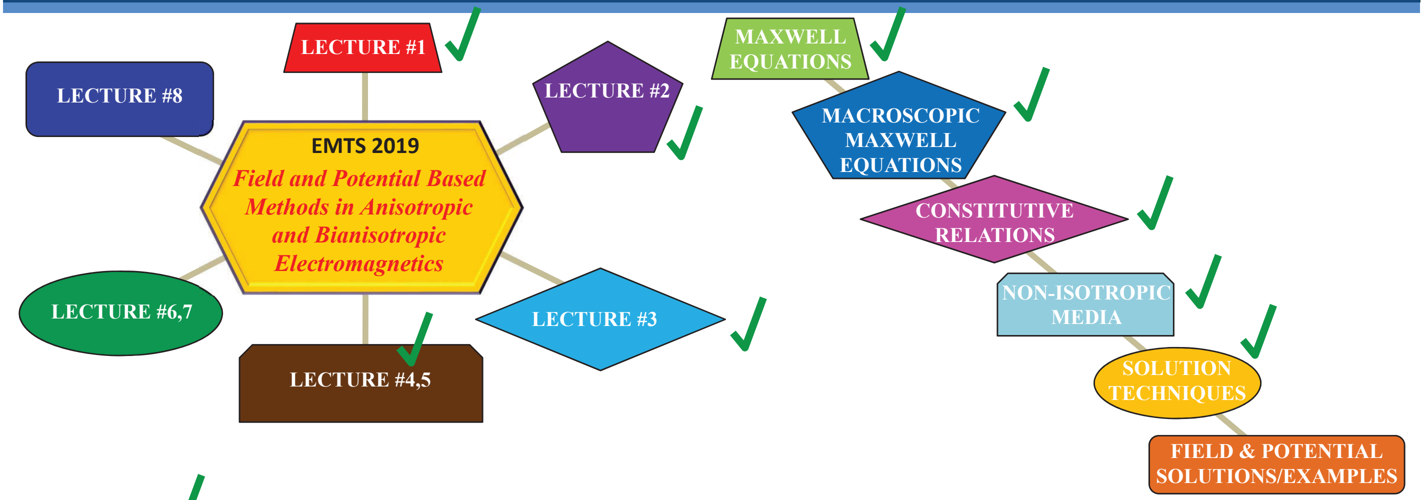
Find the Maxwell equation relations for biaxial media assuming y-invariance.

Find the plane wave normal incidence fields for bi-isotropic media.

Find the oblique $(\vec{k} = \hat{x}k_x + \hat{z}k_z)$ *plane wave fields inside a biaxial medium.*

In the kDB system, find the wave equation for the transverse B field. How is the transverse D field computed from the transverse B field?

Overview – Lectures/Big Picture



LECTURE #1: Review various forms and regimes of validity of Maxwell's equations and constitutive relations.

LECTURE #2: Discuss factors that influence anisotropy and bianisotropy.

LECTURE #3: Discuss field and potential based methods for solving electromagnetic problems.

LECTURE #4,5: Provide several examples of field based solutions in complex media within and exterior to source regions.

LECTURE #6,7: Provide several examples of potential based solutions in complex media within and exterior to source regions.

LECTURE #8: Summary, conclusions and future research.

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2019 International Symposium on Electromagnetic Theory

LECTURE #5
Field-Based Examples – Source Region

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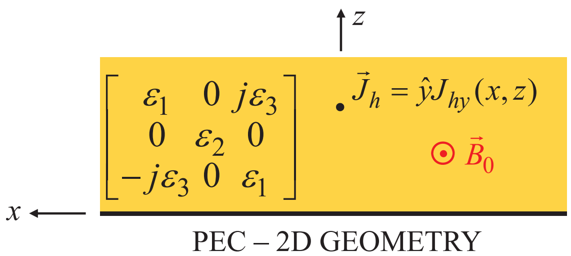
Conductor-Backed Plasma – Anisotropic Gyrotropic Media (y-invariance)



$$TE^y \ (TM^x, TM^z)$$

$$\nabla_t^2 H_y + k^2 H_y = -\hat{y}\cdot\nabla_t \times \vec{J}_{et} + j\frac{\varepsilon_3}{\varepsilon_1}\nabla_t\cdot\vec{J}_{et} + j\omega\frac{\varepsilon_1^2-\varepsilon_3^2}{\varepsilon_1}J_{hy}$$

$$\begin{bmatrix} E_x \\ E_z \end{bmatrix} = \frac{1}{\omega^2(\varepsilon_1^2-\varepsilon_3^2)}\begin{bmatrix} j\omega\varepsilon_1 & \omega\varepsilon_3 \\ -\omega\varepsilon_3 & j\omega\varepsilon_1 \end{bmatrix}\begin{bmatrix} J_{ex} + \frac{\partial H_y}{\partial z} \\ J_{ez} - \frac{\partial H_y}{\partial x} \end{bmatrix}$$

$$\nabla_t = \hat{x}\frac{\partial}{\partial x} + \hat{z}\frac{\partial}{\partial z} \ , \ k^2 = \omega^2\frac{(\varepsilon_1^2-\varepsilon_3^2)\mu_2}{\varepsilon_1}$$

$$\vec{J}_{et} = 0 \ , \ \varepsilon = \frac{\varepsilon_1^2-\varepsilon_3^2}{\varepsilon_1} \ \Rightarrow \ \frac{\partial^2 H_y(x,z)}{\partial x^2} + \frac{\partial^2 H_y(x,z)}{\partial z^2} + k^2 H_y(x,z) = j\omega\varepsilon J_{hy}(x,z) \ , \ k^2 = \omega^2\varepsilon\mu_2$$

$$\begin{bmatrix} E_x \\ E_z \end{bmatrix} = \frac{1}{\omega^2\varepsilon\varepsilon_1}\begin{bmatrix} j\omega\varepsilon_1 & \omega\varepsilon_3 \\ -\omega\varepsilon_3 & j\omega\varepsilon_1 \end{bmatrix}\begin{bmatrix} \frac{\partial H_y}{\partial z} \\ -\frac{\partial H_y}{\partial x} \end{bmatrix}$$

S. R. Seshadri, "Excitation of surface waves on a perfectly conducting screen covered with anisotropic plasma," IRE Trans. MTT, pp. 573-578, Nov. 1962.

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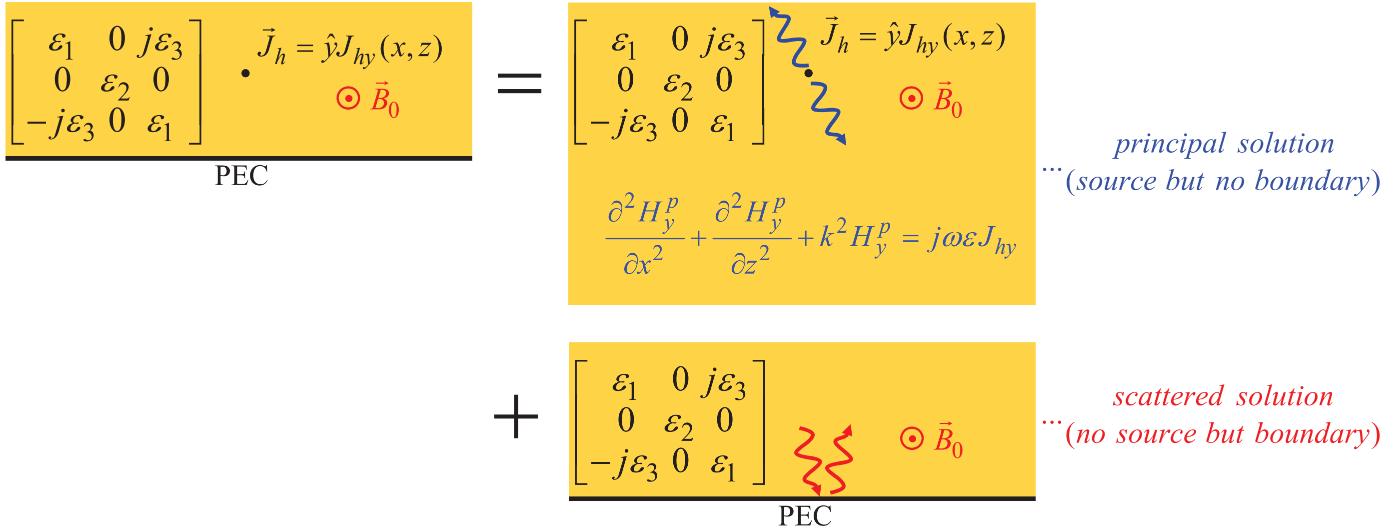
Conductor-Backed Plasma – Field Solution



Question: What ultimately justifies this solution technique?

$$\frac{\partial^2 H_y^s}{\partial x^2} + \frac{\partial^2 H_y^s}{\partial z^2} + k^2 H_y^s = 0$$

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Conductor-Backed Plasma – Principal Solution

$$f(x,z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(k_x,z)e^{jk_x x}dk_x \ , \ \tilde{f}(k_x,z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{\tilde{f}}(k_x,k_z)e^{jk_z z}dk_z \ ...generic\ Fourier\ Transform\ (FT)$$

Question: What prompts a FT approach?
Question: How does FT help?

$$\frac{\partial^2 H_y^p(x,z)}{\partial x^2} + \frac{\partial^2 H_y^p(x,z)}{\partial z^2} + k^2 H_y^p(x,z) = j\omega\varepsilon J_{hy}(x,z) \overset{FT_{xz}}{\rightarrow} \tilde{\tilde{H}}_y^p(k_x,k_z) = -\frac{j\omega\varepsilon \tilde{\tilde{J}}_{hy}(k_x,k_z)}{[k_z^2 - \underbrace{(k^2-k_x^2)}_{k_{zp}^2}]} = -\frac{j\omega\varepsilon \tilde{\tilde{J}}_{hy}(k_x,k_z)}{(k_z-k_{zp})(k_z+k_{zp})}$$

$$\tilde{H}_y^p(k_x,z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{\tilde{H}}_y^p(k_x,k_z)e^{jk_z z}dk_z \ , \ \tilde{\tilde{J}}_{hy}(k_x,k_z) = \int_{-\infty}^{\infty} \tilde{J}_{hy}(k_x,z')e^{-jk_z z'}dz' = \int_{z'} \tilde{J}_{hy}(k_x,z')e^{-jk_z z'}dz' \Rightarrow$$

$$\tilde{H}_y^p(k_x,z) = \int_{z'} \underbrace{\int_{-\infty}^{-\infty} -\frac{j\omega\varepsilon e^{jk_z z}e^{-jk_z z'}}{2\pi(k_z-k_{zp})(k_z+k_{zp})}dk_z}_{\tilde{G}_{yy}^{hh,p}(k_x,z-z')} \tilde{J}_{hy}(k_x,z')dz' = \underbrace{\int_{z'} \tilde{G}_{yy}^{hh,p}(k_x,z-z')\tilde{J}_{hy}(k_x,z')dz'}_{what\ does\ this\ represent\ physically?}$$

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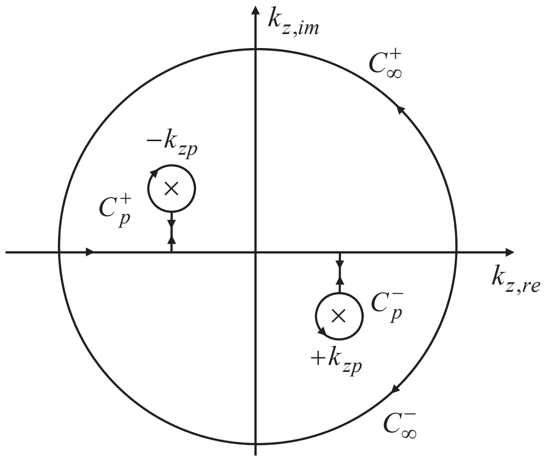
Conductor-Backed Plasma – Principal Solution



$$\tilde{G}_{yy}^{hh,p}(k_x, z-z') = \int_{-\infty}^{-\infty} -\frac{j\omega\varepsilon e^{jk_z(z-z')}}{2\pi(k_z - k_{zp})(k_z + k_{zp})} dk_z \ldots \text{simple poles at } k_z = \pm k_{zp}$$

$$e^{j(k_{z,re}+jk_{z,im})(z-z')} = e^{jk_{z,re}(z-z')} e^{-k_{z,im}(z-z')} \Rightarrow \begin{matrix} UHPC \ldots z > z' \\ LHPC \ldots z < z' \end{matrix}$$

$$\int_{-\infty}^{\infty} + \oint_{C_p^+} + \underbrace{\int_{C_\infty^+}}_{=0} = 0 \quad \Rightarrow \tilde{G}_{yy}^{hh,p} = -\frac{\omega\varepsilon e^{-jk_{zp}(z-z')}}{2k_{zp}} \ldots z > z'$$

$$\int_{-\infty}^{\infty} + \oint_{C_p^-} + \underbrace{\int_{C_\infty^-}}_{=0} = 0 \quad \Rightarrow \tilde{G}_{yy}^{hh,p} = -\frac{\omega\varepsilon e^{+jk_{zp}(z-z')}}{2k_{zp}} \ldots z < z'$$

$$\oint_C f(k_z)dk_z = 0 \ldots \text{Cauchy's Integral Theorem } (f \text{ analytic within, on } C)$$

$$\oint_C \frac{f(k_z)}{k_z - k_{z0}} dk_z = j2\pi f(k_{z0}) \ldots \text{Cauchy's Integral Formula}$$

$$\tilde{H}_y^p(k_x, z) = \int_{z'} \tilde{G}_{yy}^{hh,p}(k_x, z-z') \tilde{J}_{hy}(k_x, z')dz'$$

$$\tilde{G}_{yy}^{hh,p} = -\frac{\omega\varepsilon e^{-jk_{zp}|z-z'|}}{2k_{zp}} \ldots \text{make sense?}$$

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Conductor-Backed Plasma – Scattered and Total Solution

$$\frac{\partial^2 H_y^s(x,z)}{\partial x^2} + \frac{\partial^2 H_y^s(x,z)}{\partial z^2} + k^2 H_y^s(x,z) = 0 \overset{FT_x}{\rightarrow} \frac{\partial^2 \tilde{H}_y^s(k_x,z)}{\partial z^2} + \underbrace{(k^2 - k_x^2)}_{k_{zp}^2} \tilde{H}_y^s(k_x,z) = 0 \Rightarrow$$

$$\boxed{\tilde{H}_y^s(k_x,z) = \tilde{W}^+(k_x)e^{-jk_{zp}z} + \tilde{W}^-(k_x)e^{jk_{zp}z}} \ldots \text{scattered solution (up, down-going waves in } z)$$

$$\tilde{H}_y(k_x,z) = \tilde{H}_y^p(k_x,z) + \tilde{H}_y^s(k_x,z) = \int_{z'} \underbrace{\tilde{G}_{yy}^{hh,p}(k_x, z-z')}_{-\frac{\omega\varepsilon e^{-jk_{zp}|z-z'|}}{2k_{zp}}} \tilde{J}_{hy}(k_x,z')dz' + \tilde{W}^+(k_x)e^{-jk_{zp}z} + \tilde{W}^-(k_x)e^{jk_{zp}z} \ldots \text{total solution}$$

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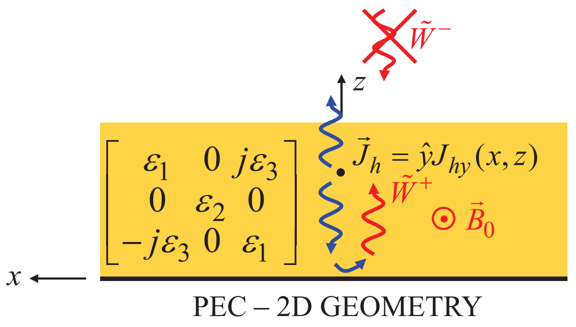
Conductor-Backed Plasma – Boundary Condition Relations

$$\tilde{H}_y(k_x,z)=\int_{z'} -\frac{\omega\varepsilon e^{-jk_{zp}|z-z'|}}{2k_{zp}}\tilde{J}_{hy}(k_x,z')dz'+\tilde{W}^+(k_x)e^{-jk_{zp}z}+\tilde{W}^-(k_x)e^{jk_{zp}z}$$

- $\tilde{H}_y(k_x,z\to\infty)\to 0 \Rightarrow \tilde{W}^-=0 \Rightarrow \tilde{H}_y(k_x,z)=\int_{z'} -\frac{\omega\varepsilon e^{-jk_{zp}|z-z'|}}{2k_{zp}}\tilde{J}_{hy}(k_x,z')dz'+\tilde{W}^+(k_x)e^{-jk_{zp}z}$



- $E_x(x,0)=0 \Rightarrow \tilde{E}_x(k_x,0)=0$ $(\Rightarrow$ *need to calculate* $\tilde{E}_x)$

Conductor-Backed Plasma – Boundary Conditions (Electric Field Calculation)

$$\tilde{H}_y(k_x,z)=\int_{z'} -\frac{\omega\varepsilon e^{-jk_{zp}|z-z'|}}{2k_{zp}}\tilde{J}_{hy}(k_x,z')dz'+\tilde{W}^+(k_x)e^{-jk_{zp}z}$$

$$E_x=\frac{1}{\omega^2\varepsilon\varepsilon_1}\left(j\omega\varepsilon_1\frac{\partial H_y}{\partial z}-\omega\varepsilon_3\frac{\partial H_y}{\partial x}\right) \Rightarrow \tilde{E}_x=\frac{1}{\omega^2\varepsilon\varepsilon_1}\left(j\omega\varepsilon_1\frac{\partial \tilde{H}_y}{\partial z}-\omega\varepsilon_3 jk_x\tilde{H}_y\right) \Rightarrow$$

$\frac{\partial}{\partial z}$ *taken in distributional sense (or use Leibnitz's rule)*

$$\tilde{E}_x(k_x,z)=\frac{1}{\omega\varepsilon\varepsilon_1}\Big[\varepsilon_1 k_{zp}\int_{z'} -\frac{\omega\varepsilon e^{-jk_{zp}|z-z'|}\,\mathrm{sgn}(z-z')}{2k_{zp}}\tilde{J}_{hy}(k_x,z')dz'$$
$$-j\varepsilon_3 k_x\int_{z'} -\frac{\omega\varepsilon e^{-jk_{zp}|z-z'|}}{2k_{zp}}\tilde{J}_{hy}(k_x,z')dz'+(\varepsilon_1 k_{zp}-j\varepsilon_3 k_x)\tilde{W}^+e^{-jk_{zp}z}\Big]$$

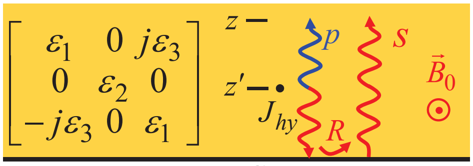
$$,\ \mathrm{sgn}(z-z')=\begin{cases}+1 \ldots z>z'\\ -1\ldots z<z'\end{cases}$$

Conductor-Backed Plasma – Boundary Condition Enforcement

$$\tilde{E}_x(k_x,0)=0 \;\Rightarrow\; -(\varepsilon_1 k_{zp}+j\varepsilon_3 k_x)\int_{z'} -\frac{\omega\varepsilon e^{-jk_{zp}z'}}{2k_{zp}}\tilde{J}_{hy}(k_x,z')dz' + (\varepsilon_1 k_{zp}-j\varepsilon_3 k_x)\tilde{W}^+ = 0 \;\Rightarrow$$

$$\tilde{W}^+ = \int_{z'} -\frac{\omega\varepsilon e^{-jk_{zp}z'}}{2k_{zp}}\frac{(\varepsilon_1 k_{zp}+j\varepsilon_3 k_x)}{(\varepsilon_1 k_{zp}-j\varepsilon_3 k_x)}\tilde{J}_{hy}(k_x,z')dz' \;\rightarrow\; \tilde{H}_y(k_x,z)=\int_{z'} -\frac{\omega\varepsilon e^{-jk_{zp}|z-z'|}}{2k_{zp}}\tilde{J}_{hy}(k_x,z')dz' + \tilde{W}^+ e^{-jk_{zp}z}$$

$$\boxed{\tilde{H}_y(k_x,z)=\int_{z'} -\frac{\omega\varepsilon}{2k_{zp}}[e^{-jk_{zp}|z-z'|}+\underbrace{\frac{\varepsilon_1 k_{zp}+j\varepsilon_3 k_x}{\varepsilon_1 k_{zp}-j\varepsilon_3 k_x}}_{R}e^{-jk_{zp}(z+z')}]\tilde{J}_{hy}(k_x,z')dz' = \int_{z'}(\underbrace{\tilde{G}_{yy}^{hh,p}+\tilde{G}_{yy}^{hh,s}}_{\tilde{G}_{yy}^{hh}})\tilde{J}_{hy}(k_x,z')dz'}$$



PEC

$$R=\frac{\varepsilon_1 k_{zp}+j\varepsilon_3 k_x}{\varepsilon_1 k_{zp}-j\varepsilon_3 k_x}=1 \ldots if\ \varepsilon_3=0\ (as\ expected)$$

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Conductor-Backed Plasma – Radiation and Surface Wave Modes

$$\boxed{\tilde{H}_y(k_x,z)=\int_{z'} -\frac{\omega\varepsilon}{2k_{zp}}[e^{-jk_{zp}|z-z'|}+\underbrace{\frac{\varepsilon_1 k_{zp}+j\varepsilon_3 k_x}{\varepsilon_1 k_{zp}-j\varepsilon_3 k_x}}_{R}e^{-jk_{zp}(z+z')}]\tilde{J}_{hy}(k_x,z')dz' = \int_{z'}(\tilde{G}_{yy}^{hh,p}+\tilde{G}_{yy}^{hh,s})\tilde{J}_{hy}(k_x,z')dz'}$$

$$H_y(x,z)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{H}_y(k_x,z)e^{jk_x x}dk_x \;,\; \tilde{J}_{hy}(k_x,z')=\int_{-\infty}^{\infty}J_{hy}(x',z')e^{-jk_x x'}dx' = \int_{x'}J_{hy}(x',z')e^{-jk_x x'}dx' \;\Rightarrow$$

$$H_y(x,z)=\int_{z'}\int_{x'}\underbrace{\int_{-\infty}^{\infty} -\frac{\omega\varepsilon}{4\pi k_{zp}}[e^{-jk_{zp}|z-z'|}+\underbrace{\frac{\varepsilon_1 k_{zp}+j\varepsilon_3 k_x}{\varepsilon_1 k_{zp}-j\varepsilon_3 k_x}}_{R}e^{-jk_{zp}(z+z')}]e^{jk_x(x-x')}dk_x}_{G_{yy}^{hh}(x-x'|z,z')}\,J_{hy}(x',z')dx'dz'$$

$k_{zp}=\sqrt{k^2-k_x^2}\ \ldots$ *branch points at* $k_x=\pm k$ *(radiation mode spectrum)*

$$\frac{\varepsilon_1 k_{zp}+j\varepsilon_3 k_x}{\varepsilon_1 k_{zp}-j\varepsilon_3 k_x}=\frac{N(k_x)}{D(k_x)}\;,$$

$N(k_x)\ldots$ *pole contribution / weight*

$D(k_x)=0\ldots$ *pole singularity (surface wave mode)*

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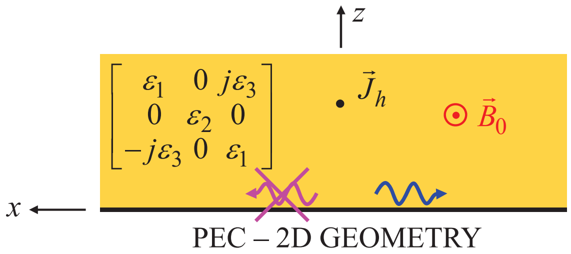
Conductor-Backed Plasma – SW (Surface Wave) Modes

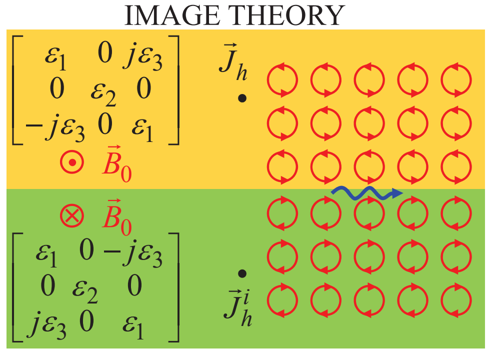
$$D(k_x)=\varepsilon_1 k_{zp} - j\varepsilon_3 k_x = 0 \;\Rightarrow\; \varepsilon_1^2 \underbrace{k_{zp}^2}_{k^2-k_x^2} = -\varepsilon_3^2 k_x^2 \;\Rightarrow\; k_x^2=\omega^2\varepsilon_1\mu_2 \;\Rightarrow\; \boxed{k_x=\pm\omega\sqrt{\varepsilon_1\mu_2}} \text{... pole singularities}$$

$$N(k_x)=\varepsilon_1 k_{zp} + j\varepsilon_3 k_x\Big|_{k_x=\pm\omega\sqrt{\varepsilon_1\mu_2}} = \begin{cases} j2\varepsilon_3\omega\sqrt{\varepsilon_1\mu_2}\,\ldots k_x=+\omega\sqrt{\varepsilon_1\mu_2} \Rightarrow e^{jk_x(x-x')}=e^{+j\omega\sqrt{\varepsilon_1\mu_2}(x-x')} \;\; SW \text{ in } -x \text{ direction} \\ 0\ldots k_x=-\omega\sqrt{\varepsilon_1\mu_2} \;\Rightarrow\; \text{no } e^{-j\omega\sqrt{\varepsilon_1\mu_2}(x-x')} \text{ so no SW in } +x \text{ direction!!!} \end{cases}$$



Why SW?
Why SW
Why no SW

IMAGE THEORY



NON-RECIPROCAL!
Lots of research occurring regarding topological insulators.

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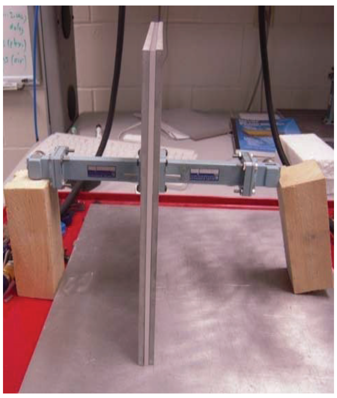
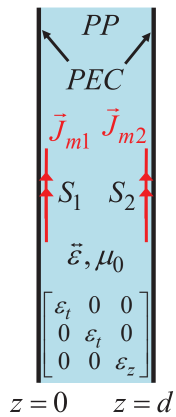
Parallel Plate Waveguide – Uniaxial Dielectric Medium (Motivation)



OBJECTIVE HERE

$$\begin{Bmatrix}\vec E(\vec r)\\ \vec H(\vec r)\end{Bmatrix} = \int_V \boxed{\begin{Bmatrix}\overline{\overline{G}}_{EH}(\vec r\,|\,\vec r\,')\\ \overline{\overline{G}}_{HH}(\vec r\,|\,\vec r\,')\end{Bmatrix}}\cdot \vec J_h(\vec r\,')\,dV'$$



$$\left.\begin{array}{l} |S_{11}^{thy}(\omega,\varepsilon_t,\varepsilon_n)-S_{11}^{exp}(\omega)|<\delta \\ |S_{21}^{thy}(\omega,\varepsilon_t,\varepsilon_n)-S_{21}^{exp}(\omega)|<\delta \end{array}\right. \Rightarrow (\varepsilon_t,\varepsilon_z)$$

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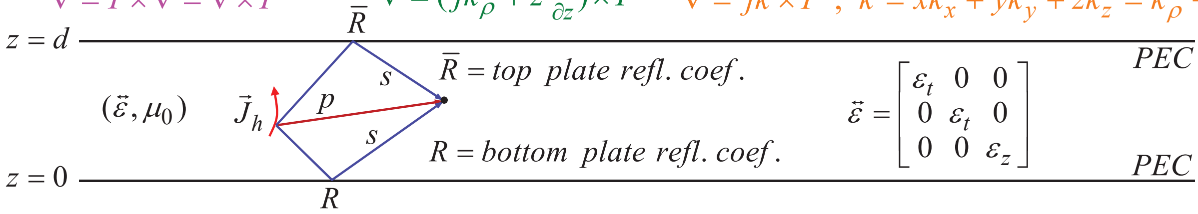
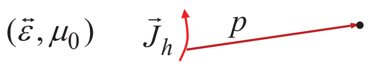
Parallel Plate Waveguide – Principal + Scattered Solutions

$\overleftrightarrow{\nabla} = \overleftrightarrow{I} \times \nabla = \nabla \times \overleftrightarrow{I}$ $\quad \overleftrightarrow{\tilde{\nabla}} = (j\vec{k}_\rho + \hat{z}\frac{\partial}{\partial z}) \times \overleftrightarrow{I}$ $\quad \overleftrightarrow{\tilde{\tilde{\nabla}}} = j\vec{k} \times \overleftrightarrow{I}$, $\vec{k} = \hat{x}k_x + \hat{y}k_y + \hat{z}k_z = \vec{k}_\rho + \hat{z}k_z$



$$\overleftrightarrow{\varepsilon} = \begin{bmatrix} \varepsilon_t & 0 & 0 \\ 0 & \varepsilon_t & 0 \\ 0 & 0 & \varepsilon_z \end{bmatrix}$$

=



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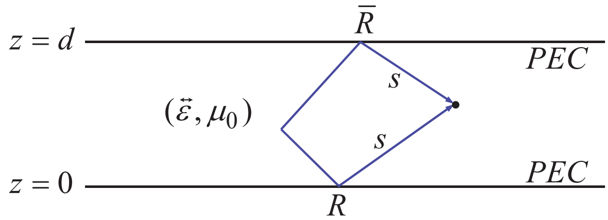


$$= \quad \overleftrightarrow{W}_E \cdot \vec{E}^p(\vec{\rho}, z) = -\overleftrightarrow{\nabla} \cdot \vec{J}_h(\vec{\rho}, z) \quad + \quad \overleftrightarrow{W}_E \cdot \vec{E}^s(\vec{\rho}, z) = 0$$

$$\overleftrightarrow{W}_E = \overleftrightarrow{\nabla} \cdot \overleftrightarrow{\nabla} - \omega^2 \mu_0 \overleftrightarrow{\varepsilon} \;,\; \overleftrightarrow{\nabla} = \nabla \times \overleftrightarrow{I}$$

Parallel Plate Waveguide – Principal Solution

$$\overleftrightarrow{\nabla} \xrightarrow{FT} \overleftrightarrow{\tilde{\tilde{\nabla}}} = j\overbrace{\vec{k}}^{\vec{k}} \times \overleftrightarrow{I} = j\vec{k} \;\Rightarrow\; \overbrace{(j\vec{k} \cdot j\vec{k} - \omega^2 \mu_0 \overleftrightarrow{\varepsilon})}^{\overleftrightarrow{\tilde{\tilde{W}}}_E} \cdot \tilde{\tilde{\vec{E}}}^p = -j\vec{k} \cdot \overbrace{\tilde{\tilde{\vec{J}}}_h}^{\hat{z}\tilde{\tilde{J}}_{zh}} \;\Rightarrow\; \tilde{\tilde{\vec{E}}}^p = -j\overleftrightarrow{\tilde{\tilde{W}}}_E^{-1} \cdot \vec{k} \cdot \tilde{\tilde{\vec{J}}}_h = \overleftrightarrow{\tilde{\tilde{G}}}{}^p_{eh} \cdot \tilde{\tilde{\vec{J}}}_h$$

$$\omega^2 \mu_0 \overleftrightarrow{\varepsilon} = \omega^2 \mu_0 [\overleftrightarrow{I}\varepsilon_t + \hat{z}\hat{z}(\varepsilon_z - \varepsilon_t)]$$

$$\overleftrightarrow{\tilde{\tilde{G}}}{}^p_{eh} = -j\overbrace{\frac{\operatorname{adj}\overleftrightarrow{\tilde{\tilde{W}}}_E}{\det \overleftrightarrow{\tilde{\tilde{W}}}_E}}^{\overleftrightarrow{\tilde{\tilde{W}}}_E^{-1}} \cdot \vec{k} = -j\frac{\overbrace{(k_z^2 - k_{zTE}^2)[-\overleftrightarrow{I}\omega^2\mu_0\varepsilon_z + \vec{k}\vec{k} + \hat{z}\hat{z}\omega^2\mu_0(\varepsilon_z - \varepsilon_t)] + (\vec{k}\times\hat{z})(\vec{k}\times\hat{z})\omega^2\mu_0(\varepsilon_z - \varepsilon_t)}^{tremendous\ work\ to\ get\ this\ result!!!}}{-\omega^2\mu_0\varepsilon_z(k_z^2 - k_{zTE}^2)(k_z^2 - k_{zTM}^2)} \cdot \vec{k}$$

$$k_{zTE}^2 = \omega^2\mu_0\varepsilon_t - k_\rho^2 \;,\; k_{zTM}^2 = k_t^2 - \tfrac{\varepsilon_t}{\varepsilon_z}k_\rho^2$$

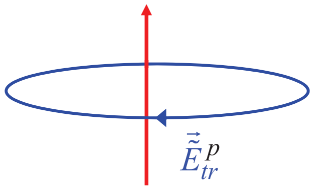
$$Note:\; \tilde{\tilde{\vec{E}}}^p = -j\overleftrightarrow{\tilde{\tilde{W}}}_E^{-1} \cdot \overbrace{\vec{k}}^{\vec{k}\times\overleftrightarrow{I}} \cdot \overbrace{\tilde{\tilde{\vec{J}}}_h}^{\hat{z}\tilde{\tilde{J}}_{hz}} = -j\overleftrightarrow{\tilde{\tilde{W}}}_E^{-1} \cdot \vec{k}\times\hat{z}\,\overbrace{\tilde{\tilde{J}}_{hz}}^{\hat{z}\cdot\hat{z}\tilde{\tilde{J}}_{hz}} = -j\overleftrightarrow{\tilde{\tilde{W}}}_E^{-1} \cdot \vec{k}\times\hat{z}\hat{z}\cdot\hat{z}\tilde{\tilde{J}}_{hz} = \overleftrightarrow{\tilde{\tilde{G}}}{}^p_{eh} \cdot \tilde{\tilde{\vec{J}}}_h$$

$$\Rightarrow \tilde{\tilde{\vec{E}}}^p = -j\frac{-\omega^2\mu_0\varepsilon_z(k_z^2 - k_{zTM}^2)\vec{k}\times\hat{z}\hat{z}}{-\omega^2\mu_0\varepsilon_z(k_z^2 - k_{zTE}^2)(k_z^2 - k_{zTM}^2)} \cdot \hat{z}\tilde{\tilde{J}}_{hz} = -\frac{j\vec{k}\times\hat{z}\hat{z}}{(k_z^2 - k_{zTE}^2)} \cdot \hat{z}\tilde{\tilde{J}}_{hz} = -\frac{j\vec{k}_\rho\times\hat{z}\hat{z}}{(k_z^2 - k_{zTE}^2)} \cdot \tilde{\tilde{\vec{J}}}_h \ldots \underbrace{no\ TM^z\ mode}_{as\ expected}$$

Parallel Plate Waveguide – Principal Solution

$$\tilde{\tilde{\vec{\vec{E}}}}{}^p_{tr}(\vec{k}_\rho,k_z) = -\frac{j\vec{k}_\rho\times\hat{z}\hat{z}}{(k_z-k_{zTE})(k_z+k_{zTE})}\cdot\hat{z}\tilde{\tilde{J}}_{hz} = \tilde{\tilde{\vec{\vec{G}}}}{}^{e,h\,p}_{tr,z}(\vec{k}_\rho,k_z)\cdot\hat{z}\tilde{\tilde{J}}_{hz}$$

$$\tilde{\vec{E}}^p_{tr}(\vec{k}_\rho,z)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{\tilde{\vec{E}}}{}^p_{tr}(\vec{k}_\rho,k_z)e^{jk_zz}dk_z = \int_0^d \underbrace{-\frac{\vec{k}_\rho\times\hat{z}\hat{z}}{2k_{zTE}}e^{-jk_{zTE}|z-z'|}}_{\tilde{\vec{\vec{G}}}{}^{e,h\,p}_{tr,z}(\vec{k}_\rho,z-z')}\cdot\tilde{\vec{J}}_{hz}(\vec{k}_\rho,z')dz'$$



Makes physical sense!

Parallel Plate Waveguide – Scattered Solution

$$\tilde{\vec{E}}^s(\vec{k}_\rho,z)=\tilde{\vec{E}}^s_0e^{jk_zz}\ \rightarrow\ \vec{\vec{W}}_E(\vec{k}_\rho,z)\cdot\tilde{\vec{E}}^s(\vec{k}_\rho,z)=0\ \Rightarrow\ \tilde{\vec{\vec{W}}}_E(\vec{k}_\rho,k_z)\cdot\tilde{\vec{E}}^s_0(\vec{k}_\rho,z)=0$$

$$\tilde{\vec{\vec{W}}}_E = j\vec{k}\cdot j\vec{k}-\omega^2\mu_0\vec{\vec{\varepsilon}} = -\overbrace{\vec{k}\times\vec{\vec{I}}\cdot(\vec{k}\times\vec{\vec{I}})}^{\vec{k}\times\vec{k}\times\vec{\vec{I}}=\vec{k}(\vec{k}\cdot\vec{\vec{I}})-\vec{\vec{I}}(\vec{k}\cdot\vec{k})}-\omega^2\mu_0\vec{\vec{\varepsilon}} = \vec{\vec{I}}\overbrace{k^2}^{k_\rho^2+k_z^2}-\vec{k}\vec{k}-\vec{\vec{I}}\omega^2\mu_0\varepsilon_t-\hat{z}\hat{z}\omega^2\mu_0(\varepsilon_z-\varepsilon_t)$$

$$\left.\tilde{\vec{\vec{W}}}_E\right|_{k_z=\mp k_{zTE}} = \tilde{\vec{\vec{W}}}{}^{\pm}_E = \vec{\vec{I}}(k_\rho^2+\underbrace{k_{zTE}^2}_{\omega^2\mu_0\varepsilon_t-k_\rho^2})-(\vec{k}_\rho\mp\hat{z}k_{zTE})(\vec{k}_\rho\mp\hat{z}k_{zTE})-\vec{\vec{I}}\omega^2\mu_0\varepsilon_t-\hat{z}\hat{z}\omega^2\mu_0(\varepsilon_z-\varepsilon_t)$$

$$=-(\vec{k}_\rho\mp\hat{z}k_{zTE})(\vec{k}_\rho\mp\hat{z}k_{zTE})-\hat{z}\hat{z}\omega^2\mu_0(\varepsilon_z-\varepsilon_t)$$

$$=-\vec{k}_\rho\vec{k}_\rho\pm\vec{k}_\rho\hat{z}k_{zTE}\pm\hat{z}k_{zTE}\vec{k}_\rho-\hat{z}\hat{z}k_{zTE}^2-\hat{z}\hat{z}\omega^2\mu_0(\varepsilon_z-\varepsilon_t)$$

$$=-\vec{k}_\rho\vec{k}_\rho\pm\vec{k}_\rho\hat{z}k_{zTE}\pm\hat{z}k_{zTE}\vec{k}_\rho+\hat{z}\hat{z}k_\rho^2-\hat{z}\hat{z}\omega^2\mu_0\varepsilon_z$$

Parallel Plate Waveguide – Scattered Solution

$$\tilde{\bar{\bar{W}}}_E^{\pm}\cdot\tilde{\vec{E}}_0^{s\pm}=[-\vec{k}_\rho\vec{k}_\rho\pm\vec{k}_\rho\hat{z}k_{zTE}\pm\hat{z}k_{zTE}\vec{k}_\rho+\hat{z}\hat{z}k_\rho^2-\hat{z}\hat{z}\omega^2\mu_0\varepsilon_z]\cdot(\tilde{\vec{E}}_{0t}^{s\pm}+\hat{z}\tilde{E}_{0z}^{s\pm})=0\;\Rightarrow$$

$$-\vec{k}_\rho\vec{k}_\rho\cdot\tilde{\vec{E}}_{0t}^{s\pm}\pm\vec{k}_\rho\hat{z}k_{zTE}\cdot\hat{z}\tilde{E}_{0z}^{s\pm}\pm\hat{z}k_{zTE}\vec{k}_\rho\cdot\tilde{\vec{E}}_{0t}^{s\pm}+\hat{z}\hat{z}k_\rho^2\cdot\hat{z}\tilde{E}_{0z}^{s\pm}-\hat{z}\hat{z}\omega^2\mu_0\varepsilon_z\cdot\hat{z}\tilde{E}_{0z}^{s\pm}=0\;\Rightarrow$$

$$-\vec{k}_\rho(\vec{k}_\rho\cdot\tilde{\vec{E}}_{0t}^{s\pm}\mp k_{zTE}\tilde{E}_{0z}^{s\pm})=0\;,\;\hat{z}(\pm k_{zTE}\vec{k}_\rho\cdot\tilde{\vec{E}}_{0t}^{s\pm}+k_\rho^2\tilde{E}_{0z}^{s\pm}-\omega^2\mu_0\varepsilon_z\tilde{E}_{0z}^{s\pm})=0$$

$$\therefore\;\begin{array}{l}\vec{k}_\rho\cdot\tilde{\vec{E}}_{0t}^{s\pm}\mp k_{zTE}\tilde{E}_{0z}^{s\pm}=0\;\;or\;\;\vec{k}_\rho\cdot\tilde{\vec{E}}_{0t}^{s\pm}=\pm k_{zTE}\tilde{E}_{0z}^{s\pm}\;\;(a)\\ \pm k_{zTE}\vec{k}_\rho\cdot\tilde{\vec{E}}_{0t}^{s\pm}+(k_\rho^2-\omega^2\mu_0\varepsilon_z)\tilde{E}_{0z}^{s\pm}=0\;\;(b)\end{array}$$

$$(a)\rightarrow(b)\Rightarrow\pm k_{zTE}\overbrace{\vec{k}_\rho\cdot\tilde{\vec{E}}_{0t}^{s\pm}}^{\pm k_{zTE}\tilde{E}_{0z}^{s\pm}}+(k_\rho^2-\omega^2\mu_0\varepsilon_z)\tilde{E}_{0z}^{s\pm}=0\Rightarrow\overbrace{k_{zTE}^2}^{\omega^2\mu_0\varepsilon_t-k_\rho^2}\tilde{E}_{0z}^{s\pm}+(k_\rho^2-\omega^2\mu_0\varepsilon_z)\tilde{E}_{0z}^{s\pm}=0$$

$$\Rightarrow\omega^2\mu_0(\varepsilon_t-\varepsilon_z)\tilde{E}_{0z}^{s\pm}=0\Rightarrow\boxed{\tilde{E}_{0z}^{s\pm}=0\ldots since\;\varepsilon_t\neq\varepsilon_z\;(as\;expected\;for\;a\;TE^z\;mode)}$$

Parallel Plate Waveguide – Scattered and Total Solution

$$\tilde{E}_{0z}^{s\pm}=0\rightarrow\vec{k}_\rho\cdot\tilde{\vec{E}}_{0t}^{s\pm}=\pm k_{zTE}\tilde{E}_{0z}^{s\pm}\Rightarrow\vec{k}_\rho\cdot\tilde{\vec{E}}_{0t}^{s\pm}=0\;\;or\;\;\tilde{E}_{0y}^{s\pm}=-\frac{k_x}{k_y}\tilde{E}_{0x}^{s\pm}\Rightarrow$$

$$\tilde{\vec{E}}_{0t}^{s\pm}=\hat{x}\tilde{E}_{0x}^{s\pm}+\hat{y}\tilde{E}_{0y}^{s\pm}=(\hat{x}-\hat{y}\frac{k_x}{k_y})\tilde{E}_{0x}^{s\pm}=\frac{1}{k_y}\overbrace{(\hat{x}k_y-\hat{y}k_x)}^{\vec{k}_\rho\times\hat{z}}\tilde{E}_{0x}^{s\pm}=\boxed{\vec{k}_\rho\times\hat{z}\frac{\tilde{E}_{0x}^{s\pm}}{k_y}=\tilde{\vec{E}}_{0tr}^{s\pm}}$$

$$\therefore\;\boxed{\tilde{\vec{E}}_{tr}^{s}=\vec{k}_\rho\times\hat{z}\frac{\tilde{E}_{0x}^{s+}}{k_y}e^{-jk_{zTE}z}+\vec{k}_\rho\times\hat{z}\frac{\tilde{E}_{0x}^{s-}}{k_y}e^{+jk_{zTE}z}}\ldots\begin{array}{c}scattered\;\;solution\\(very\;\;tedious\;\;journey)!\end{array}$$

$$\tilde{\vec{E}}_{tr}=\tilde{\vec{E}}_{tr}^{p}+\tilde{\vec{E}}_{tr}^{s}=\int_0^d-\frac{\vec{k}_\rho\times\hat{z}\hat{z}}{2k_{zTE}}e^{-jk_{zTE}|z-z'|}\cdot\tilde{\vec{J}}_{hz}(\vec{k}_\rho,z')dz'+\vec{k}_\rho\times\hat{z}\frac{\tilde{E}_{0x}^{s+}}{k_y}e^{-jk_{zTE}z}+\vec{k}_\rho\times\hat{z}\frac{\tilde{E}_{0x}^{s-}}{k_y}e^{+jk_{zTE}z}$$

Parallel Plate Waveguide – Boundary Conditions

$$\tilde{\vec{E}}_{tr} = \tilde{\vec{E}}^{p}_{tr} + \tilde{\vec{E}}^{s}_{tr} = \int_0^d -\frac{\vec{k}_\rho \times \hat{z}\hat{z}}{2k_{zTE}} e^{-jk_{zTE}|z-z'|} \cdot \tilde{\vec{J}}_{hz}(\vec{k}_\rho, z')dz' + \vec{k}_\rho \times \hat{z}\frac{\tilde{E}^{s+}_{0x}}{k_y} e^{-jk_{zTE}z} + \vec{k}_\rho \times \hat{z}\frac{\tilde{E}^{s-}_{0x}}{k_y} e^{+jk_{zTE}z}$$

- $\tilde{\vec{E}}_{tr}(z=0) = 0 \;\Rightarrow\; \vec{k}_\rho \times \hat{z}\left[\overbrace{\int_0^d -\frac{e^{-jk_{zTE}z'}}{2k_{zTE}} \tilde{J}_{hz}(\vec{k}_\rho, z')dz'}^{V^-/k_y} + \frac{\tilde{E}^{s+}_{0x}}{k_y} + \frac{\tilde{E}^{s-}_{0x}}{k_y}\right] = 0 \;\Rightarrow\; \boxed{\tilde{E}^{s+}_{0x} = RV^- + R\tilde{E}^{s-}_{0x}\ ,\ R=-1}$

- $\tilde{\vec{E}}_{tr}(z=d) = 0 \;\Rightarrow\; \vec{k}_\rho \times \hat{z}e^{-jk_{zTE}d}\left[\overbrace{\int_0^d -\frac{e^{jk_{zTE}z'}}{2k_{zTE}} \tilde{J}_{hz}(\vec{k}_\rho, z')dz'}^{V^+/k_y} + \frac{\tilde{E}^{s+}_{0x}}{k_y} + \frac{\tilde{E}^{s-}_{0x}}{k_y}e^{j2k_{zTE}d}\right] = 0$

$$\Rightarrow \boxed{\tilde{E}^{s-}_{0x} = \bar{R}V^+ e^{-j2k_{zTE}d} + \bar{R}\tilde{E}^{+}_{0x} e^{-j2k_{zTE}d}\ ,\ \bar{R}=-1}$$

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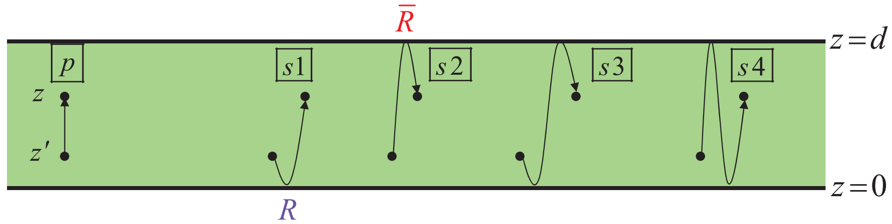
Parallel Plate Waveguide – Solution

$$\Rightarrow \tilde{E}^{s+}_{0x} = \frac{RV^- + R\bar{R}V^+ e^{-j2k_{zTE}d}}{1-R\bar{R}e^{-j2k_{zTE}d}}\ ,\ \tilde{E}^{s-}_{0x} = \frac{\bar{R}V^+ e^{-j2k_{zTE}d} + R\bar{R}V^- e^{-j2k_{zTE}d}}{1-R\bar{R}e^{-j2k_{zTE}d}}\ ,\ R=\bar{R}=-1$$

$$\tilde{\vec{E}}_{tr} = \int_0^d -\frac{\vec{k}_\rho \times \hat{z}\hat{z}}{2k_{zTE}}\Big[\overbrace{e^{-jk_{zTE}|z-z'|}}^{\boxed{p}} + \frac{\overbrace{Re^{-jk_{zTE}(z+z')}}^{\boxed{s1}} + \overbrace{\bar{R}e^{-jk_{zTE}(2d-z-z')}}^{\boxed{s2}} + \overbrace{R\bar{R}e^{-jk_{zTE}(2d-z+z')}}^{\boxed{s3}} + \overbrace{R\bar{R}e^{-jk_{zTE}(2d+z-z')}}^{\boxed{s4}}}{1-R\bar{R}e^{-j2k_{zTE}d}}\Big] \cdot \tilde{\vec{J}}_{hz}(\vec{k}_\rho, z')dz'$$

Prove this result.



$$\boxed{\tilde{\vec{E}}_{tr} = \int_0^d j\vec{k}_\rho \times \hat{z}\frac{\cos k_{zTE}[d-|z-z'|] - \cos k_{zTE}[d-(z+z')]}{2k_{zTE}\sin k_{zTE}d}\hat{z}\cdot \tilde{\vec{J}}_{hz}(\vec{k}_\rho, z')dz' = \int_0^d \tilde{\vec{G}}^{e,h}_{tr,z}(\vec{k}_\rho, z\,|\,z') \cdot \tilde{\vec{J}}_{hz}(\vec{k}_\rho, z')dz'}$$

Does this result make sense?

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Field Based Examples (Source Region) – Key Take-Aways!

KEY Take-Aways

Use EM theorems to aid in analysis and physical insight.

Fourier transforms and complex analysis are critical tools in EM!

Solving problems directly with Maxwell equations can be challenging!!

Field Based Examples (Source Region) – Homework

Work through the details of the conductor-backed plasma analysis.

Are surface waves expected for the conductor-backed plasma under z-bias?

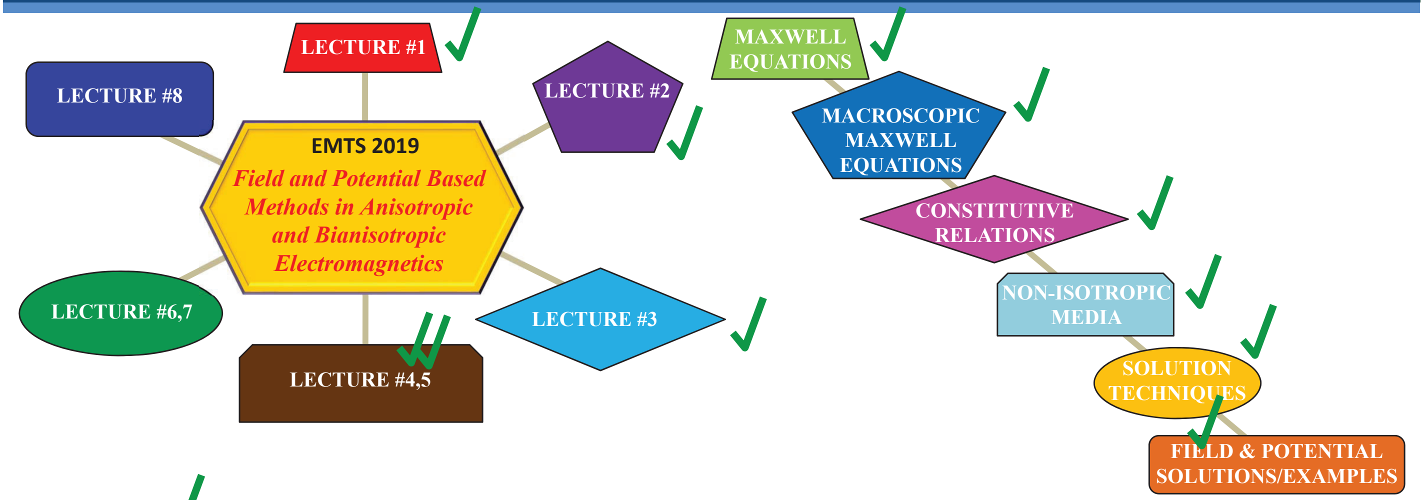
Work through the details of the parallel-plate waveguide analysis.

Find the pole singularities (surface mode k_z's) of the parallel-plate waveguide.

In the parallel-plate waveguide, would one expect a radiation-mode spectrum to exist? Can you find an easy way to show it does not exist?

Find an expression for the parallel – plate spatial domain field $\vec{E}_{tr}(\vec{\rho}, z)$

Overview – Lectures/Big Picture



LECTURE #1: Review various forms and regimes of validity of Maxwell's equations and constitutive relations.

LECTURE #2: Discuss factors that influence anisotropy and bianisotropy.

LECTURE #3: Discuss field and potential based methods for solving electromagnetic problems.

LECTURE #4,5: Provide several examples of field based solutions in complex media within and exterior to source regions.

LECTURE #6,7: Provide several examples of potential based solutions in complex media within and exterior to source regions.

LECTURE #8: Summary, conclusions and future research.

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LECTURE #6
Potential-Based Examples – Source Free Region

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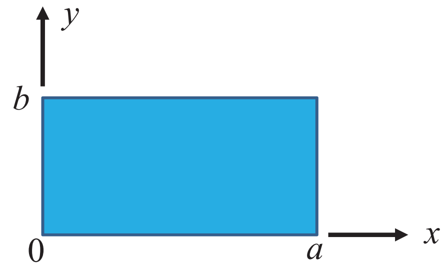
Rectangular Waveguide – Anisotropic Uniaxial Media (TM^z Modes)



Rectangular Waveguide (PEC Walls, Source-free uniaxial region)

$$\ddot{\varepsilon} = \begin{bmatrix} \varepsilon_t & 0 & 0 \\ 0 & \varepsilon_t & 0 \\ 0 & 0 & \varepsilon_z \end{bmatrix}, \ddot{\mu} = \begin{bmatrix} \mu_t & 0 & 0 \\ 0 & \mu_t & 0 \\ 0 & 0 & \mu_z \end{bmatrix}, \ddot{\xi} = \ddot{\zeta} = 0, s_1 = s_2 = 0$$

$$\begin{bmatrix} L_1 & L_2 \\ L_3 & L_4 \end{bmatrix} \begin{bmatrix} \psi \\ \theta \end{bmatrix} = \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}$$

$$L_1 = -\frac{\mu_z \Delta_t}{\mu_t \Delta_z} \nabla_t^2 - \frac{\Delta_t}{\mu_t} \frac{\partial}{\partial z} (\frac{\mu_t}{\Delta_t} \frac{\partial}{\partial z}) - \frac{\omega \Delta_t}{\mu_t} \frac{\partial}{\partial z} (\frac{\mu_t \xi_g - \mu_g \xi_t}{\Delta_t}) + \omega (\frac{\mu_t \zeta_g - \mu_g \zeta_t}{\mu_t}) \frac{\partial}{\partial z} - \omega^2 [\Delta_t - \frac{\mu_g (\varepsilon_t \mu_g - \xi_g \zeta_t) + \zeta_g (\mu_t \xi_g - \mu_g \xi_t)}{\mu_t}]$$

$$L_2 = -\frac{\xi_z \Delta_t}{\mu_t \Delta_z} \nabla_t^2 - \frac{\Delta_t}{\mu_t} \frac{\partial}{\partial z} (\frac{\xi_t}{\Delta_t} \frac{\partial}{\partial z}) - \frac{\omega \Delta_t}{\mu_t} \frac{\partial}{\partial z} (\frac{\varepsilon_g \mu_t - \xi_t \zeta_g}{\Delta_t}) - \omega (\frac{\varepsilon_t \mu_g - \xi_t \zeta_g}{\mu_t}) \frac{\partial}{\partial z} - \omega^2 [\frac{\zeta_t \Delta_t}{\mu_t} - \frac{\mu_g (\varepsilon_t \zeta_g - \varepsilon_g \zeta_t) + \zeta_g (\varepsilon_g \mu_t - \xi_t \zeta_g)}{\mu_t}]$$

$$L_3 = -\frac{\zeta_z \Delta_t}{\varepsilon_t \Delta_z} \nabla_t^2 - \frac{\Delta_t}{\varepsilon_t} \frac{\partial}{\partial z} (\frac{\zeta_t}{\Delta_t} \frac{\partial}{\partial z}) + \frac{\omega \Delta_t}{\varepsilon_t} \frac{\partial}{\partial z} (\frac{\varepsilon_t \mu_g - \xi_g \zeta_t}{\Delta_t}) + \omega (\frac{\varepsilon_g \mu_t - \xi_g \zeta_t}{\varepsilon_t}) \frac{\partial}{\partial z} - \omega^2 [\frac{\xi_t \Delta_t}{\varepsilon_t} - \frac{\varepsilon_g (\mu_t \xi_g - \mu_g \xi_t) + \xi_g (\varepsilon_t \mu_g - \xi_g \zeta_t)}{\varepsilon_t}]$$

$$L_4 = -\frac{\varepsilon_z \Delta_t}{\varepsilon_t \Delta_z} \nabla_t^2 - \frac{\Delta_t}{\varepsilon_t} \frac{\partial}{\partial z} (\frac{\varepsilon_t}{\Delta_t} \frac{\partial}{\partial z}) + \frac{\omega \Delta_t}{\varepsilon_t} \frac{\partial}{\partial z} (\frac{\varepsilon_t \zeta_g - \varepsilon_g \zeta_t}{\Delta_t}) - \omega (\frac{\varepsilon_t \xi_g - \varepsilon_g \xi_t}{\varepsilon_t}) \frac{\partial}{\partial z} - \omega^2 [\Delta_t - \frac{\varepsilon_g (\varepsilon_g \mu_t - \xi_t \zeta_g) + \xi_g (\varepsilon_t \zeta_g - \varepsilon_g \zeta_t)}{\varepsilon_t}]$$

$$s_1 = -\frac{\Delta_t}{\mu_t} \frac{\partial}{\partial z} (\frac{\mu_t}{\Delta_t} u_e) + \omega (\frac{\mu_t \zeta_g - \mu_g \zeta_t}{\mu_t}) u_e + \frac{\mu_z \Delta_t}{\mu_t \Delta_z} J_{ez} + \frac{\Delta_t}{\mu_t} \frac{\partial}{\partial z} (\frac{\xi_t}{\Delta_t} u_h) + \omega (\frac{\varepsilon_t \mu_g - \xi_t \zeta_g}{\mu_t}) u_h - \frac{j\omega \Delta_t}{\mu_t} v_h - \frac{\xi_z \Delta_t}{\mu_t \Delta_z} J_{hz}$$

$$s_2 = -\frac{\Delta_t}{\varepsilon_t} \frac{\partial}{\partial z} (\frac{\zeta_t}{\Delta_t} u_e) + \omega (\frac{\varepsilon_g \mu_t - \xi_g \zeta_t}{\varepsilon_t}) u_e - \frac{j\omega \Delta_t}{\varepsilon_t} v_e + \frac{\zeta_z \Delta_t}{\varepsilon_t \Delta_z} J_{ez} + \frac{\Delta_t}{\varepsilon_t} \frac{\partial}{\partial z} (\frac{\varepsilon_t}{\Delta_t} u_h) + \omega (\frac{\varepsilon_t \xi_g - \varepsilon_g \xi_t}{\varepsilon_t}) u_h - \frac{\varepsilon_z \Delta_t}{\varepsilon_t \Delta_z} J_{hz}$$

Show this.

$$\begin{bmatrix} \Phi \\ \pi \end{bmatrix} = \frac{1}{j\omega \Delta_t} \begin{bmatrix} \omega (\mu_t \xi_g - \mu_g \xi_t) \psi + \omega (\varepsilon_g \mu_t - \xi_t \zeta_g) \theta + \mu_t (\frac{\partial \psi}{\partial z} - u_e) + \xi_t (\frac{\partial \theta}{\partial z} + u_h) \\ \omega (\varepsilon_t \mu_g - \xi_g \zeta_t) \psi + \omega (\varepsilon_t \zeta_g - \varepsilon_g \zeta_t) \theta - \zeta_t (\frac{\partial \psi}{\partial z} - u_e) - \varepsilon_t (\frac{\partial \theta}{\partial z} + u_h) \end{bmatrix}$$

$$\vec{E}_t = \nabla_t \Phi + \nabla_t \times \hat{z}\theta, \quad \vec{H}_t = \nabla_t \pi + \nabla_t \times \hat{z}\psi, \quad \vec{E}_z = -\hat{z} \frac{\mu_z}{j\omega \Delta_z} \nabla_t^2 \psi - \hat{z} \frac{\xi_z}{j\omega \Delta_z} \nabla_t^2 \theta - \frac{\hat{z}\mu_z \hat{z}}{j\omega \Delta_z} \cdot \vec{J}_e + \frac{\hat{z}\xi_z \hat{z}}{j\omega \Delta_z} \cdot \vec{J}_h$$

$$\vec{H}_z = \hat{z} \frac{\varepsilon_z}{j\omega \Delta_z} \nabla_t^2 \theta + \hat{z} \frac{\zeta_z}{j\omega \Delta_z} \nabla_t^2 \psi - \frac{\hat{z}\varepsilon_z \hat{z}}{j\omega \Delta_z} \cdot \vec{J}_h + \frac{\hat{z}\zeta_z \hat{z}}{j\omega \Delta_z} \cdot \vec{J}_e$$

TM^z Modes

$$(\frac{\varepsilon_t}{\varepsilon_z} \nabla_t^2 + \frac{\partial^2}{\partial z^2} + \omega^2 \varepsilon_t \mu_t) \psi = 0, \quad \Phi = \frac{1}{j\omega \varepsilon_t} \frac{\partial \psi}{\partial z}$$

$$\vec{E}_t = \nabla_t \Phi, \quad \vec{E}_z = -\hat{z} \frac{1}{j\omega \varepsilon_z} \nabla_t^2 \psi, \quad \vec{H}_t = \nabla_t \times \hat{z}\psi$$

Rectangular Waveguide – Separation of Variables and Boundary Condition Relations

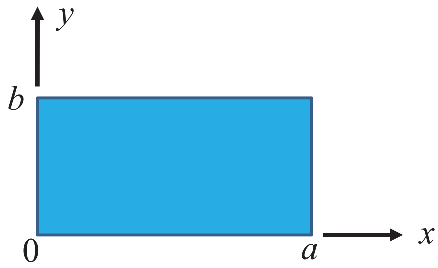
$$\psi = f(x)g(y)h(z) \rightarrow \left(\frac{\varepsilon_t}{\varepsilon_z}\overbrace{\nabla_t^2}^{\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}} + \frac{\partial^2}{\partial z^2} + \overbrace{\omega^2\varepsilon_t\mu_t}^{k_t^2}\right)\psi(x,y,z) = 0 \Rightarrow \frac{\varepsilon_t}{\varepsilon_z} gh\frac{d^2 f}{dx^2} + \frac{\varepsilon_t}{\varepsilon_z} fh\frac{d^2 g}{dy^2} + fg\frac{d^2 h}{dz^2} = -k_t^2 fgh$$

$$\Rightarrow \frac{\varepsilon_t}{\varepsilon_z}\underbrace{\frac{1}{f}\frac{d^2 f}{dx^2}}_{-k_x^2} + \frac{\varepsilon_t}{\varepsilon_z}\underbrace{\frac{1}{g}\frac{d^2 g}{dy^2}}_{-k_y^2} + \underbrace{\frac{1}{h}\frac{d^2 h}{dz^2}}_{-k_z^2} = -k_t^2 \quad \therefore$$

Why pick sin,cos in x and y and exponentials in z?

$$\psi = \overbrace{(A\sin k_x x + B\cos k_x x)(C\sin k_y y + D\cos k_y y)(Ee^{-jk_z z} + Fe^{jk_z z})}$$

$$\frac{\varepsilon_t}{\varepsilon_z}(k_x^2 + k_y^2) + k_z^2 = k_t^2 \;\; or \;\; k_z = \sqrt{k_t^2 - \frac{\varepsilon_t}{\varepsilon_z}(k_x^2 + k_y^2)}$$



Rectangular Waveguide (PEC Walls, Source-free uniaxial region)

$$\vec{E}_t = \nabla_t \Phi = \nabla_t\left(\frac{1}{j\omega\varepsilon_t}\frac{\partial\psi}{\partial z}\right) = \frac{1}{j\omega\varepsilon_t}\left(\hat{x}\frac{\partial^2\psi}{\partial x\partial z} + \hat{y}\frac{\partial^2\psi}{\partial y\partial z}\right)$$

$$\vec{E}_z = -\hat{z}\frac{1}{j\omega\varepsilon_z}\nabla_t^2\psi = \hat{z}\frac{k_x^2+k_y^2}{j\omega\varepsilon_z}\psi$$

$$E_{y,z}(x=0,a) = 0 \;\forall\, y,z \;\Rightarrow\; \psi(x=0,a) = 0 \;\forall\, y,z$$

$$E_{x,z}(y=0,b) = 0 \;\forall\, x,z \;\Rightarrow\; \psi(y=0,b) = 0 \;\forall\, x,z$$

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Rectangular Waveguide – Boundary Condition Enforcement

$$\psi = (A\sin k_x x + B\cos k_x x)(C\sin k_y y + D\cos k_y y)(Ee^{-jk_z z} + Fe^{jk_z z})$$

- $\psi(x=0) = 0 \;\forall\, y,z \;\Rightarrow\; B(C\sin k_y y + D\cos k_y y)(Ee^{-jk_z z} + Fe^{jk_z z}) = 0 \;\forall\, y,z$ $\boxed{\therefore\; B = 0}$
- $\psi(x=a) = 0 \;\forall\, y,z \;\Rightarrow\; A\sin \underbrace{k_x a}_{m\pi}(C\sin k_y y + D\cos k_y y)(Ee^{-jk_z z} + Fe^{jk_z z}) = 0 \;\forall\, y,z$ $\boxed{\therefore\; k_{xm} = \frac{m\pi}{a} \ldots m = 1,2,3,\ldots}$

  What does k_{xm} describe physically?

$$\psi = \sin k_{xm}x(C\sin k_y y + D\cos k_y y)(Ee^{-jk_z z} + Fe^{jk_z z})$$

- $\psi(y=0) = 0 \;\forall\, x,z \;\Rightarrow\; D\sin k_{xm}x(Ee^{-jk_z z} + Fe^{jk_z z}) = 0 \;\forall\, x,z$ $\boxed{\therefore\; D = 0}$
- $\psi(y=b) = 0 \;\forall\, x,z \;\Rightarrow\; C\sin k_{xm}x \sin \underbrace{k_y b}_{n\pi}(Ee^{-jk_z z} + Fe^{jk_z z}) = 0 \;\forall\, x,z$ $\boxed{\therefore\; k_{yn} = \frac{n\pi}{b} \ldots n = 1,2,3,\ldots}$

  What does k_{yn} describe physically?

$$\boxed{\psi_{mn} = \sin k_{xm}x\sin k_{yn}y(\tilde{A}_{mn}^{+}e^{-jk_{zmn}z} - \tilde{A}_{mn}^{-}e^{jk_{zmn}z}) \;,\; k_{zmn} = \underbrace{\sqrt{k_t^2 - \frac{\varepsilon_t}{\varepsilon_z}(k_{xm}^2 + k_{yn}^2)} = \sqrt{k_t^2 - \frac{\varepsilon_t}{\varepsilon_z}k_{cmn}^2}}}$$

Note: material properties and boundaries affect the allowed propagation factor k_{zmn}

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Rectangular Waveguide – Field Calculation

$$\psi_{mn} = \sin k_{xm}x \sin k_{yn}y(\tilde{A}_{mn}^{+}e^{-jk_{zmn}z} - \tilde{A}_{mn}^{-}e^{jk_{zmn}z}) \ , \ k_{zmn} = \sqrt{k_t^2 - \frac{\varepsilon_t}{\varepsilon_z}(k_{xm}^2 + k_{yn}^2)} = \sqrt{k_t^2 - \frac{\varepsilon_t}{\varepsilon_z}k_{cmn}^2}$$

$$\vec{E}_t = \nabla_t \Phi = \vec{E}_{t\ell} = \hat{x}\frac{1}{j\omega\varepsilon_t}\frac{\partial^2\psi}{\partial x \partial z} + \hat{y}\frac{1}{j\omega\varepsilon_t}\frac{\partial^2\psi}{\partial y \partial z}$$

$$= -\frac{k_{zmn}}{\omega\varepsilon_t}\overbrace{(\hat{x}k_{xm}\cos k_{xm}x \sin k_{yn}y + \hat{y}k_{yn}\sin k_{xm}x \cos k_{yn}y)}^{\vec{e}_{tmn}}(\tilde{A}_{mn}^{+}e^{-jk_{zmn}z} + \tilde{A}_{mn}^{-}e^{jk_{zmn}z})$$

$$= \vec{e}_{tmn}(A_{mn}^{+}e^{-jk_{zmn}z} + A_{mn}^{-}e^{jk_{zmn}z}) \ , \ A_{mn}^{\pm} = -\frac{k_{zmn}}{\omega\varepsilon_t}\tilde{A}_{mn}^{\pm}$$

$$\vec{E}_z = -\frac{\nabla_t^2\psi}{j\omega\varepsilon_z} = \hat{z}\frac{j\varepsilon_t k_{cmn}^2}{\varepsilon_z k_{zmn}}\sin k_{xm}x \sin k_{yn}y(A_{mn}^{+}e^{-jk_{zmn}z} - A_{mn}^{-}e^{jk_{zmn}z})$$

$$\vec{H}_t = \nabla_t \times \hat{z}\psi = \vec{H}_{tr} = \vec{h}_{tmn}(A_{mn}^{+}e^{-jk_{zmn}z} - A_{mn}^{-}e^{jk_{zmn}z}) \ , \ \vec{h}_{tmn} = \frac{\hat{z}\times\vec{e}_{tmn}}{Z_{mn}} \ , \ Z_{mn} = \frac{k_{zmn}}{\omega\varepsilon_t}$$

Physical nature of the field clearly revealed using the potential-based method.

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Parallel Plate Waveguide – Two Layer Uniaxial Media (TE^Z Modes)

PEC — $z=d$

$$k_{z2} = \sqrt{\omega^2\varepsilon_{t2}\mu_{t2} - \frac{\mu_{t2}}{\mu_{z2}}k_{\rho 2}^2}$$

$\bar{\varepsilon}_2, \bar{\mu}_2$:

$$(\frac{\mu_{t2}}{\mu_{z2}}\nabla_t^2 + \frac{\partial^2}{\partial z^2} + \omega^2\varepsilon_{t2}\mu_{t2})\theta_2 = 0 \ , \ \pi_2 = -\frac{1}{j\omega\mu_{t2}}\frac{\partial\theta_2}{\partial z}$$

$$\vec{E}_{t2} = \nabla_t \times \hat{z}\theta_2 \ , \ \vec{H}_{t2} = \nabla_t\pi_2 \ , \ \vec{H}_{z2} = \hat{z}\frac{1}{j\omega\mu_{z2}}\nabla_t^2\theta_2$$

TE^Z

$z=0$

$$k_{z1} = \sqrt{\omega^2\varepsilon_{t1}\mu_{t1} - \frac{\mu_{t1}}{\mu_{z1}}k_{\rho 1}^2}$$

$\bar{\varepsilon}_1, \bar{\mu}_1$:

$$(\frac{\mu_{t1}}{\mu_{z1}}\nabla_t^2 + \frac{\partial^2}{\partial z^2} + \omega^2\varepsilon_{t1}\mu_{t1})\theta_1 = 0 \ , \ \pi_1 = -\frac{1}{j\omega\mu_{t1}}\frac{\partial\theta_1}{\partial z}$$

$$\vec{E}_{t1} = \nabla_t \times \hat{z}\theta_1 \ , \ \vec{H}_{t1} = \nabla_t\pi_1 \ , \ \vec{H}_{z1} = \hat{z}\frac{1}{j\omega\mu_{z1}}\nabla_t^2\theta_1$$

TE^Z

PEC — $z=-h$

$\phi -$ *invariant, source – free structure*

$$\left.\begin{aligned}\theta_1(\rho,z) &= (\tilde{A}_1 e^{-jk_{\rho 1}\rho} + \tilde{B}_1 e^{jk_{\rho 1}\rho})(\tilde{C}_1\sin k_{z1}z + \tilde{D}_1\cos k_{z1}z)\\ \theta_2(\rho,z) &= (\tilde{A}_2 e^{-jk_{\rho 2}\rho} + \tilde{B}_2 e^{jk_{\rho 2}\rho})(\tilde{C}_2\sin k_{z2}z + \tilde{D}_2\cos k_{z2}z)\end{aligned}\right\}\cdots \text{ via separation of variables}$$

$$\underbrace{\theta_{1,2}(\rho\to\infty,z)\to 0}_{\text{no ingoing waves from } \infty} \Rightarrow \tilde{B}_1, \tilde{B}_2 = 0 \Rightarrow \boxed{\begin{aligned}\theta_1(\rho,z) &= e^{-jk_{\rho 1}\rho}[A_1\sin k_{z1}(z+h) + B_1\cos k_{z1}(z+h)]\\ \theta_2(\rho,z) &= e^{-jk_{\rho 2}\rho}[A_2\sin k_{z2}(z-d) + B_2\cos k_{z2}(z-d)]\end{aligned}}$$

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Parallel Plate Waveguide – PEC Boundary Conditions

$\theta_2(d)=0$ — PEC — $z=d$

$\vec{\vec{\varepsilon}}_2,\vec{\vec{\mu}}_2$

$$(\frac{\mu_{t2}}{\mu_{z2}}\nabla_t^2+\frac{\partial^2}{\partial z^2}+\omega^2\varepsilon_{t2}\mu_{t2})\theta_2=0\ ,\ \pi_2=-\frac{1}{j\omega\mu_{t2}}\frac{\partial\theta_2}{\partial z}$$

$$\vec{E}_{t2}=\nabla_t\times\hat{z}\theta_2\ ,\ \vec{H}_{t2}=\nabla_t\pi_2\ ,\ \vec{H}_{z2}=\hat{z}\frac{1}{j\omega\mu_{z2}}\nabla_t^2\theta_2$$

$z=0$

$\vec{\vec{\varepsilon}}_1,\vec{\vec{\mu}}_1$

$$(\frac{\mu_{t1}}{\mu_{z1}}\nabla_t^2+\frac{\partial^2}{\partial z^2}+\omega^2\varepsilon_{t1}\mu_{t1})\theta_1=0\ ,\ \pi_1=-\frac{1}{j\omega\mu_{t1}}\frac{\partial\theta_1}{\partial z}$$

$$\vec{E}_{t1}=\nabla_t\times\hat{z}\theta_1\ ,\ \vec{H}_{t1}=\nabla_t\pi_1\ ,\ \vec{H}_{z1}=\hat{z}\frac{1}{j\omega\mu_{z1}}\nabla_t^2\theta_1$$

$\theta_1(-h)=0$ — PEC — $z=-h$

$$\theta_1(\rho,z)=e^{-jk_{\rho 1}\rho}[A_1\sin k_{z1}(z+h)+B_1\cos k_{z1}(z+h)]$$
$$\theta_2(\rho,z)=e^{-jk_{\rho 2}\rho}[A_2\sin k_{z2}(z-d)+B_2\cos k_{z2}(z-d)]$$

- $\theta_1(-h)=0\ \forall\rho\ \Rightarrow\ B_1=0\ \Rightarrow\ \theta_1(\rho,z)=A_1e^{-jk_{\rho 1}\rho}\sin k_{z1}(z+h)$
- $\theta_2(d)=0\ \forall\rho\ \Rightarrow\ B_2=0\ \Rightarrow\ \theta_2(\rho,z)=A_2e^{-jk_{\rho 2}\rho}\sin k_{z2}(z-d)$

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Parallel Plate Waveguide – Material Interface Boundary Conditions

PEC — $z=d$

$\vec{\vec{\varepsilon}}_2,\vec{\vec{\mu}}_2$

$$(\frac{\mu_{t2}}{\mu_{z2}}\nabla_t^2+\frac{\partial^2}{\partial z^2}+\omega^2\varepsilon_{t2}\mu_{t2})\theta_2=0\ ,\ \pi_2=-\frac{1}{j\omega\mu_{t2}}\frac{\partial\theta_2}{\partial z}$$

$$\vec{E}_{t2}=\nabla_t\times\hat{z}\theta_2\ ,\ \vec{H}_{t2}=\nabla_t\pi_2\ ,\ \vec{H}_{z2}=\hat{z}\frac{1}{j\omega\mu_{z2}}\nabla_t^2\theta_2$$

$\theta_1(0^-)=\theta_2(0^+)$
$\pi_1(0^-)=\pi_2(0^+)$ — $z=0$

$\vec{\vec{\varepsilon}}_1,\vec{\vec{\mu}}_1$

$$(\frac{\mu_{t1}}{\mu_{z1}}\nabla_t^2+\frac{\partial^2}{\partial z^2}+\omega^2\varepsilon_{t1}\mu_{t1})\theta_1=0\ ,\ \pi_1=-\frac{1}{j\omega\mu_{t1}}\frac{\partial\theta_1}{\partial z}$$

$$\vec{E}_{t1}=\nabla_t\times\hat{z}\theta_1\ ,\ \vec{H}_{t1}=\nabla_t\pi_1\ ,\ \vec{H}_{z1}=\hat{z}\frac{1}{j\omega\mu_{z1}}\nabla_t^2\theta_1$$

PEC — $z=-h$

$$\theta_1=A_1e^{-jk_{\rho 1}\rho}\sin k_{z1}(z+h)\ ,\ \pi_1=-\frac{1}{j\omega\mu_{t1}}\frac{\partial\theta_1}{\partial z}=A_1jZ_1^{-1}e^{-jk_{\rho 1}\rho}\cos k_{z1}(z+h)\ ,\ Z_1=\frac{\omega\mu_{t1}}{k_{z1}}$$

$$\theta_2=A_2e^{-jk_{\rho 2}\rho}\sin k_{z2}(z-d)\ ,\ \pi_2=-\frac{1}{j\omega\mu_{t2}}\frac{\partial\theta_2}{\partial z}=A_2jZ_2^{-1}e^{-jk_{\rho 2}\rho}\cos k_{z2}(z-d)\ ,\ Z_2=\frac{\omega\mu_{t2}}{k_{z2}}$$

- $\theta_1(0^-)=\theta_2(0^+)\ \forall\rho\Rightarrow \overbrace{k_{\rho 1}=k_{\rho 2}}^{\text{make sense?}}=k_\rho\ ,\ A_1\sin k_{z1}h=-A_2\sin k_{z2}d$ ...*continuity of tangential* $\vec{E}$
- $\pi_1(0^-)=\pi_2(0^+)\ \forall\rho\Rightarrow k_{\rho 1}=k_{\rho 2}=k_\rho\ ,\ A_1Z_1^{-1}\cos k_{z1}h=A_2Z_2^{-1}\cos k_{z2}d$ ...*continuity of tangential* $\vec{H}$

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Parallel Plate Waveguide – Scalar Potential Summary

$$\left.\begin{aligned} A_1 \sin k_{z1}h &= -A_2 \sin k_{z2}d \quad (1) \\ A_1 Z_1^{-1} \cos k_{z1}h &= A_2 Z_2^{-1} \cos k_{z2}d \quad (2) \end{aligned}\right\} (1) \div (2) \;\Rightarrow\; Z_1 \tan k_{z1}h = -Z_2 \tan k_{z2}d \;\; or$$

$$\boxed{Z_1 \tan k_{z1}h + Z_2 \tan k_{z2}d = 0} \;\ldots \text{transcendental equation for } k_\rho \;;\; k_{z1} = \sqrt{\omega^2 \varepsilon_{t1}\mu_{t1} - \tfrac{\mu_{t1}}{\mu_{z1}} k_\rho^2} \;,\; k_{z2} = \sqrt{\omega^2 \varepsilon_{t2}\mu_{t2} - \tfrac{\mu_{t2}}{\mu_{z2}} k_\rho^2}$$

$$\theta_1 = A_1 e^{-jk_\rho \rho} \sin k_{z1}(z+h)$$

$$\pi_1 = A_1 j Z_1^{-1} e^{-jk_\rho \rho} \cos k_{z1}(z+h)$$

$$\theta_2 = A_2 e^{-jk_\rho \rho} \sin k_{z2}(z-d) = -A_1 e^{-jk_\rho \rho} \frac{\sin k_{z1}h}{\sin k_{z2}d} \sin k_{z2}(z-d) \;\; \ldots using\ (1)$$

$$\pi_2 = A_2 j Z_2^{-1} e^{-jk_\rho \rho} \cos k_{z2}(z-d) = A_1 j Z_1^{-1} e^{-jk_\rho \rho} \frac{\cos k_{z1}h}{\cos k_{z2}d} \cos k_{z2}(z-d) \;\; \ldots using\ (2)$$

All boundary conditions satisfied! ✓

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Parallel Plate Waveguide – Field Calculation/Summary

$$\vec{E}_{t1} = \underbrace{\nabla_t}_{\hat{\rho}\frac{\partial}{\partial \rho}} \times \hat{z}\theta_1 = -\hat{\phi}\frac{\partial \theta_1}{\partial \rho} = \hat{\phi} A_1 j k_\rho e^{-jk_\rho \rho} \sin k_{z1}(z+h)$$

$$\vec{H}_{t1} = \nabla_t \pi_1 = \hat{\rho}\frac{\partial \pi_1}{\partial \rho} = \hat{\rho} A_1 k_\rho Z_1^{-1} e^{-jk_\rho \rho} \cos k_{z1}(z+h)$$

$$\vec{H}_{z1} = \hat{z}\frac{1}{j\omega\mu_{z1}} \underbrace{\nabla_t^2}_{\frac{\partial^2}{\partial \rho^2}} \theta_1 = -\hat{z} A_1 \frac{k_\rho^2}{j\omega\mu_{z1}} e^{-jk_\rho \rho} \sin k_{z1}(z+h)$$

$$\vec{B}_{z1} = \mu_{z1}\vec{H}_{z1} = -\hat{z} A_1 \frac{k_\rho^2}{j\omega} e^{-jk_\rho \rho} \sin k_{z1}(z+h)$$

$$\vec{E}_{t2} = \nabla_t \times \hat{z}\theta_2 = -\hat{\phi} j k_\rho A_1 e^{-jk_\rho \rho} \frac{\sin k_{z1}h}{\sin k_{z2}d} \sin k_{z2}(z-d)$$

$$\vec{H}_{t2} = \nabla_t \pi_2 = \hat{\rho} A_1 k_\rho Z_1^{-1} e^{-jk_\rho \rho} \frac{\cos k_{z1}h}{\cos k_{z2}d} \cos k_{z2}(z-d)$$

$$\vec{H}_{z2} = \hat{z}\frac{1}{j\omega\mu_{z2}} \nabla_t^2 \theta_2 = \hat{z} A_1 \frac{k_\rho^2}{j\omega\mu_{z2}} e^{-jk_\rho \rho} \frac{\sin k_{z1}h}{\sin k_{z2}d} \sin k_{z2}(z-d)$$

$$\vec{B}_{z2} = \mu_{z2}\vec{H}_{z2} = \hat{z} A_1 \frac{k_\rho^2}{j\omega} e^{-jk_\rho \rho} \frac{\sin k_{z1}h}{\sin k_{z2}d} \sin k_{z2}(z-d)$$

$$\boxed{Z_1 \tan k_{z1}h + Z_2 \tan k_{z2}d = 0} \;,\; k_{z1} = \sqrt{\omega^2 \varepsilon_{t1}\mu_{t1} - \tfrac{\mu_{t1}}{\mu_{z1}} k_\rho^2} \;,\; k_{z2} = \sqrt{\omega^2 \varepsilon_{t2}\mu_{t2} - \tfrac{\mu_{t2}}{\mu_{z2}} k_\rho^2} \;,\; Z_1 = \frac{\omega\mu_{t1}}{k_{z1}} \;,\; Z_2 = \frac{\omega\mu_{t2}}{k_{z2}}$$

Note: $\vec{B}_{z1}(-h) = 0$, $\vec{B}_{z2}(d) = 0$, $B_{z1}(0^-) = B_{z2}(0^+)$ ... *normal boundary conditions on $\vec{B}$ satisfied as expected!*

What is the physical nature of the transverse fields?

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Potential Based Examples (Source Free Region) – Key Take-Aways!

KEY Take-Aways

Scalar potentials can simplify the mathematical formulation.

Scalar potentials can enhance physical insight!

Scalar potentials are limited to bianisotropic gyrotropic media.

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Potential Based Examples (Source Free Region) – Homework

Work through the TM^Z mode details of the uniaxial rectangular waveguide.

Find the TE^Z modes of a uniaxial filled rectangular waveguide.

Work through the TE^Z mode details of the two-layer uniaxial parallel-plate waveguide analysis.

Find the TM^Z modes for the two-layer uniaxial parallel-plate waveguide.

Find the modes that can exist in a z-invariant parallel-plate waveguide filled with a z-biased anisotropic gyrotropic media.

Find the TE^Y and TM^Y modes that can exist in a source-free y-invariant rectangular waveguide filled with a y-biased anisotropic gyrotropic media.

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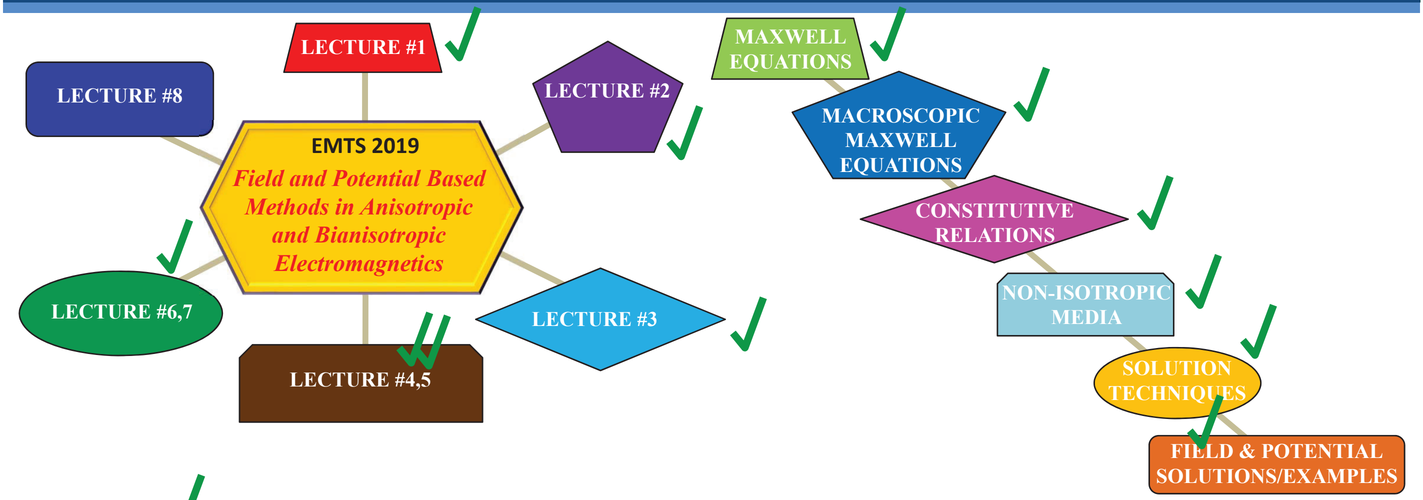
Overview – Lectures/Big Picture



LECTURE #1: Review various forms and regimes of validity of Maxwell's equations and constitutive relations.
LECTURE #2: Discuss factors that influence anisotropy and bianisotropy.
LECTURE #3: Discuss field and potential based methods for solving electromagnetic problems.
LECTURE #4,5: Provide several examples of field based solutions in complex media within and exterior to source regions.
LECTURE #6,7: Provide several examples of potential based solutions in complex media within and exterior to source regions.
LECTURE #8: Summary, conclusions and future research.

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2019 International Symposium on Electromagnetic Theory

LECTURE #7
Potential-Based Examples – Source Region

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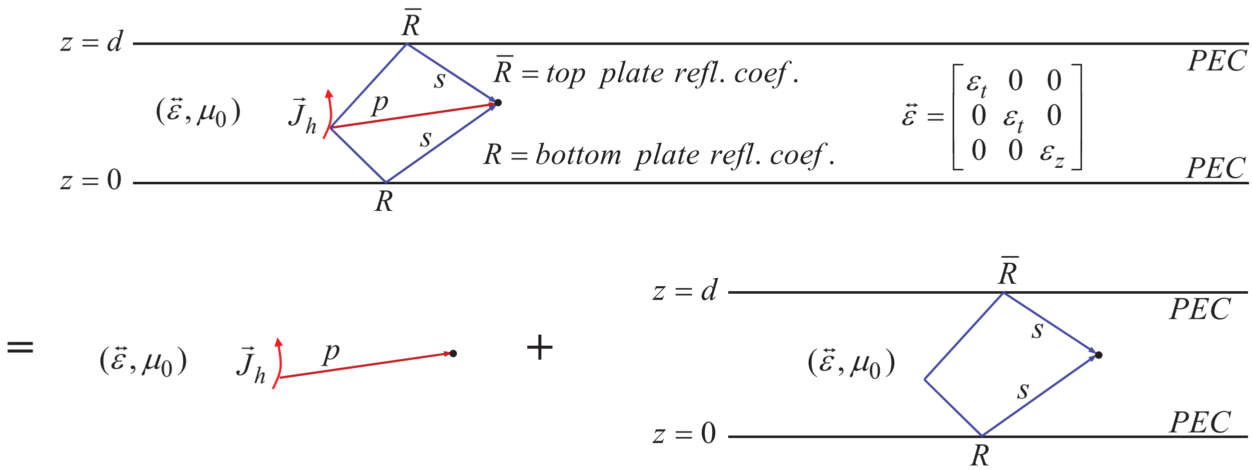
Parallel Plate Waveguide – Principal + Scattered Solutions (TE^z Modes)



$$\overleftrightarrow{\varepsilon} = \begin{bmatrix} \varepsilon_t & 0 & 0 \\ 0 & \varepsilon_t & 0 \\ 0 & 0 & \varepsilon_z \end{bmatrix}$$

$$= (-\nabla_t^2 - \frac{\partial^2}{\partial z^2} - \omega^2 \varepsilon_t \mu_0)\theta^p(\vec{\rho}, z) = -J_{hz}(\vec{\rho}, z) \quad + \quad (-\nabla_t^2 - \frac{\partial^2}{\partial z^2} - \omega^2 \varepsilon_t \mu_0)\theta^s(\vec{\rho}, z) = 0$$

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Parallel Plate Waveguide – Principal Solution

$$f(\vec{\rho},z)=\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tilde{f}(\vec{k}_\rho,z)e^{j\vec{k}_\rho\cdot\vec{\rho}}\underbrace{d^2k_\rho}_{dk_xdk_y}\ ,\ \tilde{f}(\vec{k}_\rho,z)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{\tilde{f}}(\vec{k}_\rho,k_z)e^{jk_zz}dk_z\ ...\ \begin{matrix}generic\ Fourier\ transforms\\ \vec{\rho}=\hat{x}x+\hat{y}y\ ,\ \vec{k}_\rho=\hat{x}k_x+\hat{y}k_y\end{matrix}$$

$$(-\nabla_t^2-\frac{\partial^2}{\partial z^2}-\omega^2\varepsilon_t\mu_0)\theta^p(\vec{\rho},z)=-J_{hz}(\vec{\rho},z)\ \xrightarrow{FT_{\vec{\rho},z}}\ [k_z^2-(\underbrace{\omega^2\varepsilon_t\mu_0-k_\rho^2}_{k_{zTE}^2})]\tilde{\tilde{\theta}}^p(\vec{k}_\rho,k_z)=-\tilde{\tilde{J}}_{hz}(\vec{k}_\rho,k_z)$$

$$\therefore\ \tilde{\tilde{\theta}}^p(\vec{k}_\rho,k_z)=-\frac{1}{(k_z-k_{zTE})(k_z+k_{zTE})}\tilde{\tilde{J}}_{hz}(\vec{k}_\rho,k_z)=\tilde{\tilde{G}}^p_{\theta h}(\vec{k}_\rho,k_z)\tilde{\tilde{J}}_{hz}(\vec{k}_\rho,k_z)$$

$$\tilde{\theta}^p(\vec{k}_\rho,z)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{\tilde{\theta}}^p(\vec{k}_\rho,k_z)e^{jk_zz}dk_z\ ,\tilde{\tilde{J}}_{hz}(\vec{k}_\rho,k_z)=\int_{-\infty}^{\infty}\tilde{J}_{hz}(\vec{k}_\rho,z')e^{-jk_zz'}dz'=\int_0^d\tilde{J}_{hz}(\vec{k}_\rho,z')e^{-jk_zz'}dz'\Rightarrow$$

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Parallel Plate Waveguide – Principal Solution

$$\tilde{\theta}^p(\vec{k}_\rho,z)=\int_0^d\int_{-\infty}^{\infty}-\frac{e^{jk_z(z-z')}}{2\pi(k_z-k_{zTE})(k_z+k_{zTE})}dk_z\tilde{J}_{hz}(\vec{k}_\rho,z')dz'=\int_0^d\tilde{G}^p_{\theta h}(\vec{k}_\rho,z-z')\tilde{J}_{hz}(\vec{k}_\rho,z')dz'$$

$$\tilde{G}^p_{\theta h}(\vec{k}_\rho,z-z')=\int_{-\infty}^{\infty}-\frac{e^{jk_z(z-z')}}{2\pi(k_z-k_{zTE})(k_z+k_{zTE})}dk_z=\frac{je^{-jk_{zTE}|z-z'|}}{2k_{zTE}}\ ...using\ complex\ plane\ analysis$$

$$\therefore\ \boxed{\tilde{\theta}^p(\vec{k}_\rho,z)=\int_0^d\tilde{G}^p_{\theta h}(\vec{k}_\rho,z-z')\tilde{J}_{hz}(\vec{k}_\rho,z')dz'=\int_0^d\frac{je^{-jk_{zTE}|z-z'|}}{2k_{zTE}}\tilde{J}_{hz}(\vec{k}_\rho,z')dz'}$$

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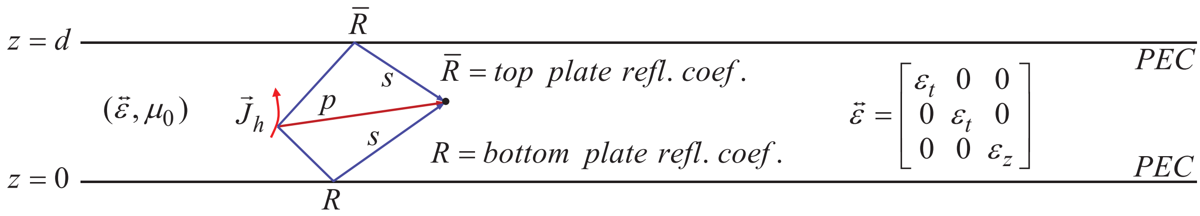
Parallel Plate Waveguide – Scattered and Total Solutions

$$(-\nabla_t^2 - \frac{\partial^2}{\partial z^2} - \omega^2 \varepsilon_t \mu_0)\theta^s(\vec{\rho}, z) = 0 \overset{FT_{\vec{\rho}}}{\rightarrow} (k_\rho^2 - \frac{\partial^2}{\partial z^2} - \omega^2 \varepsilon_t \mu_0]\tilde{\theta}^s(\vec{k}_\rho, z) = 0 \Rightarrow$$

$$\boxed{\tilde{\theta}^s(\vec{k}_\rho, z) = \tilde{W}^+(\vec{k}_\rho)e^{-jk_{zTE}z} + \tilde{W}^-(\vec{k}_\rho)e^{jk_{zTE}z}}$$

$$\boxed{\tilde{\theta}(\vec{k}_\rho, z) = \tilde{\theta}^p(\vec{k}_\rho, z) + \tilde{\theta}^s(\vec{k}_\rho, z) = \int_0^d \frac{je^{-jk_{zTE}|z-z'|}}{2k_{zTE}} \tilde{J}_{hz}(\vec{k}_\rho, z')dz' + \tilde{W}^+(\vec{k}_\rho)e^{-jk_{zTE}z} + \tilde{W}^-(\vec{k}_\rho)e^{jk_{zTE}z}}$$

Parallel Plate Waveguide – Boundary Condition Relations



$$\bar{\bar{\varepsilon}} = \begin{bmatrix} \varepsilon_t & 0 & 0 \\ 0 & \varepsilon_t & 0 \\ 0 & 0 & \varepsilon_z \end{bmatrix}$$

$\tilde{\theta}(z = d) = 0$

$\tilde{\theta}(z = 0) = 0$

$$\vec{E}_t(\vec{\rho}, z) = \nabla_t \times \hat{z}\theta(\vec{\rho}, z) \overset{FT_{\vec{\rho}}}{\rightarrow} \tilde{\vec{E}}_t(\vec{k}_\rho, z) = j\vec{k}_\rho \times \hat{z}\tilde{\theta}(\vec{k}_\rho, z)$$

$$\tilde{\vec{E}}_t(z = 0, d) = 0 \Rightarrow \boxed{\tilde{\theta}(z = 0, d) = 0 \ldots boundary\ conditions\ at\ the\ PEC's}$$

Parallel Plate Waveguide – Boundary Condition Enforcement

$$\tilde{\theta}(\vec{k}_\rho,z)=\tilde{\theta}^p(\vec{k}_\rho,z)+\tilde{\theta}^s(\vec{k}_\rho,z)=\int_0^d \frac{je^{-jk_{zTE}|z-z'|}}{2k_{zTE}}\tilde{J}_{hz}(\vec{k}_\rho,z')dz'+\tilde{W}^+(\vec{k}_\rho)e^{-jk_{zTE}z}+\tilde{W}^-(\vec{k}_\rho)e^{jk_{zTE}z}$$

- $\tilde{\theta}(z=0)=0 \;\Rightarrow\; \overbrace{\int_0^d \frac{je^{-jk_{zTE}z'}}{2k_{zTE}}\tilde{J}_{hz}(\vec{k}_\rho,z')dz'}^{\tilde{V}^-}+\tilde{W}^+ +\tilde{W}^- =0 \;\Rightarrow\; \boxed{\tilde{W}^+ = R\tilde{V}^- + R\tilde{W}^-\;,\; R=-1}$

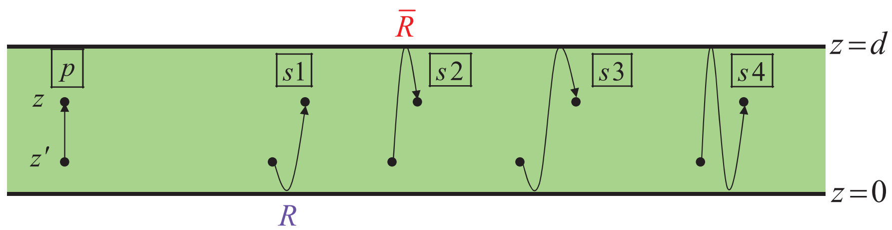
$$\tilde{\theta}(z=d)=0 \;\Rightarrow\; e^{-jk_{zTE}d}\overbrace{\int_0^d \frac{je^{jk_{zTE}z'}}{2k_{zTE}}\tilde{J}_{hz}(\vec{k}_\rho,z')dz'}^{\tilde{V}^+}+\tilde{W}^+e^{-jk_{zTE}d}+\tilde{W}^-e^{jk_{zTE}d}=0 \;\Rightarrow$$

$$\boxed{\tilde{W}^- = \bar{R}\tilde{V}^+e^{-j2k_{zTE}d}+\bar{R}\tilde{W}^+e^{-j2k_{zTE}d}\;,\;\bar{R}=-1}$$

Parallel Plate Waveguide – Potential Solution

$$\tilde{W}^+ = \frac{R\tilde{V}^- + R\bar{R}\tilde{V}^+e^{-j2k_{zTE}d}}{1-R\bar{R}e^{-j2k_{zTE}d}}\;,\;\tilde{W}^- = \frac{\bar{R}\tilde{V}^+e^{-j2k_{zTE}d}+R\bar{R}\tilde{V}^-e^{-j2k_{zTE}d}}{1-R\bar{R}e^{-j2k_{zTE}d}}\;,\; R=\bar{R}=-1$$

$$\tilde{\theta}(\vec{k}_\rho,z)=\int_0^d \frac{j}{2k_{zTE}}\Big[\overbrace{e^{-jk_{zTE}|z-z'|}}^{\boxed{p}}+\frac{\overbrace{Re^{-jk_{zTE}(z+z')}}^{\boxed{s1}}+\overbrace{\bar{R}e^{-jk_{zTE}(2d-z-z')}}^{\boxed{s2}}+\overbrace{R\bar{R}e^{-jk_{zTE}(2d-z+z')}}^{\boxed{s3}}+\overbrace{R\bar{R}e^{-jk_{zTE}(2d+z-z')}}^{\boxed{s4}}}{1-R\bar{R}e^{-j2k_{zTE}d}}\Big]\tilde{J}_{hz}(\vec{k}_\rho,z')dz'$$



$$\boxed{\tilde{\theta}(\vec{k}_\rho,z)=\int_0^d \frac{\cos k_{zTE}[d-|z-z'|]-\cos k_{zTE}[d-(z+z')]}{2k_{zTE}\sin k_{zTE}d}\tilde{J}_{hz}(\vec{k}_\rho,z')dz'=\int_0^d \tilde{G}_{\theta h}(\vec{k}_\rho,z\,|\,z')\tilde{J}_{hz}(\vec{k}_\rho,z')dz'}$$

Parallel Plate Waveguide – Electric Field Solution

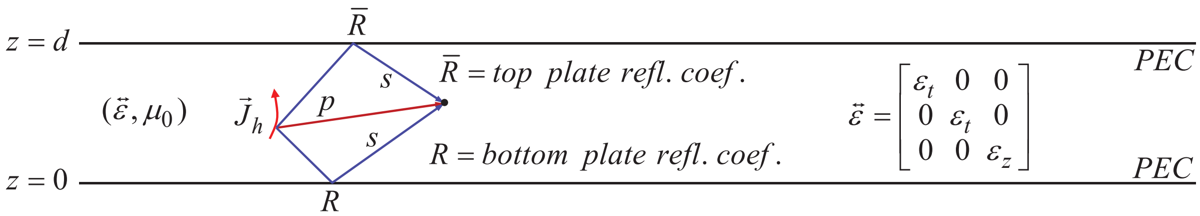
$$\tilde{\theta}(\vec{k}_\rho, z) = \int_0^d \frac{\cos k_{zTE}[d - |z - z'|] - \cos k_{zTE}[d - (z + z')]}{2k_{zTE}\sin k_{zTE} d} \tilde{J}_{hz}(\vec{k}_\rho, z')dz' = \int_0^d \tilde{G}_{\theta h}(\vec{k}_\rho, z \mid z')\tilde{J}_{hz}(\vec{k}_\rho, z')dz'$$

$$\tilde{\vec{E}}_t(\vec{k}_\rho, z) = \tilde{\vec{E}}_{tr}(\vec{k}_\rho, z) = j\vec{k}_\rho \times \hat{z}\tilde{\theta}(\vec{k}_\rho, z) \ , \ \tilde{J}_{hz} = \hat{z}\cdot\tilde{\vec{J}}_h \ \Rightarrow$$

$$\tilde{\vec{E}}_{tr}(\vec{k}_\rho, z) = \int_0^d \underbrace{j\vec{k}_\rho \times \hat{z} \frac{\cos k_{zTE}[d - |z - z'|] - \cos k_{zTE}[d - (z + z')]}{2k_{zTE}\sin k_{zTE} d}\hat{z}}_{\tilde{\overline{\overline{G}}}^{e,h}_{tr,z}} \cdot \tilde{\vec{J}}_h(\vec{k}_\rho, z')dz' \ldots$$

in agreement with field based solution

Parallel Plate Waveguide – Physical Understanding



$\overline{R} = top\ plate\ refl.\ coef.$

$R = bottom\ plate\ refl.\ coef.$

$$\overline{\overline{\varepsilon}} = \begin{bmatrix} \varepsilon_t & 0 & 0 \\ 0 & \varepsilon_t & 0 \\ 0 & 0 & \varepsilon_z \end{bmatrix}$$

$$\tilde{\vec{E}}_{tr}(\vec{k}_\rho, z) = \int_0^d \overbrace{j\vec{k}_\rho \times \hat{z}\frac{\cos k_{zTE}[d - |z - z'|] - \cos k_{zTE}[d - (z + z')]}{2k_{zTE}\sin k_{zTE} d}\hat{z}}^{\tilde{\overline{\overline{G}}}^{e,h}_{tr,z}} \cdot \tilde{\vec{J}}_h(\vec{k}_\rho, z')dz'$$

expected standing waves in z

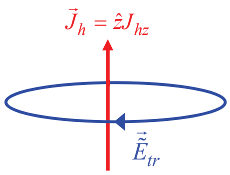
maintains transverse rotational electric field

z magnetic current

poles are parallel plate modes

$$k_{zTE}d = m\pi \Rightarrow k_{zTE}^2 = \frac{m^2\pi^2}{d^2} \Rightarrow$$

$$\omega^2\varepsilon_t\mu_0 - k_\rho^2 = \frac{m^2\pi^2}{d^2} \Rightarrow k_\rho = \sqrt{\omega^2\varepsilon_t\mu_0 - \frac{m^2\pi^2}{d^2}}$$



$$-\nabla \times \vec{E} = \vec{J}_h + \frac{\partial \vec{B}}{\partial t}$$

Parallel Plate Waveguide – Spatial Domain Field

$$\tilde{\vec{E}}_{tr}(\vec{k}_\rho, z) = \int_0^d \overbrace{j\vec{k}_\rho \times \hat{z} \frac{\cos k_{zTE}[d - |z - z'|] - \cos k_{zTE}[d - (z + z')]}{2k_{zTE} \sin k_{zTE} d}}^{\tilde{\bar{G}}^{e,h}_{tr,z}} \hat{z} \cdot \tilde{\vec{J}}_h(\vec{k}_\rho, z') dz'$$

$$\vec{E}_{tr}(\vec{\rho}, z) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{\vec{E}}_{tr}(\vec{k}_\rho, z) e^{j\vec{k}_\rho \cdot \vec{\rho}} d^2 k_\rho \ , \ \tilde{\vec{J}}_h(\vec{k}_\rho, z') = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \vec{J}_h(\vec{\rho}', z') e^{-j\vec{k}_\rho \cdot \vec{\rho}'} d^2 \rho' = \int_S \vec{J}_h(\vec{\rho}', z') e^{-j\vec{k}_\rho \cdot \vec{\rho}'} dS' \Rightarrow$$

$$\boxed{\vec{E}_{tr}(\vec{\rho}, z) = \iint_S \int_0^d \underbrace{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} j\vec{k}_\rho \times \hat{z} \frac{\cos k_{zTE}[d - |z - z'|] - \cos k_{zTE}[d - (z + z')] e^{j\vec{k}_\rho \cdot (\vec{\rho} - \vec{\rho}')} d^2 k_\rho}{4\pi^2 \, 2k_{zTE} \sin k_{zTE} d}}_{G^{e,h}_{tr,z}(\vec{\rho} - \vec{\rho}', z|z')} \hat{z} \cdot \vec{J}_h(\vec{\rho}', z') dz' dS'}$$

Potential Based Examples (Source Region) – Key Take-Aways!

KEY Take-Aways

Scalar potentials can simplify the mathematical formulation.

Scalar potentials can enhance physical insight!

Scalar potentials are limited to bianisotropic gyrotropic media.

Potential Based Examples (Source Region) – Homework

Work through the TE^Z mode details of the uniaxial parallel-plate waveguide.

Find the remaining fields of the uniaxial parallel-plate waveguide.

Find the TM^Z mode Green function from the uniaxial parallel-plate waveguide analysis.

Find the TE^Z Green functions of a PEC-backed uniaxial slab waveguide having a z-directed electric current.

Find the Green functions for a magnetic current immersed in a bi-isotropic medium.

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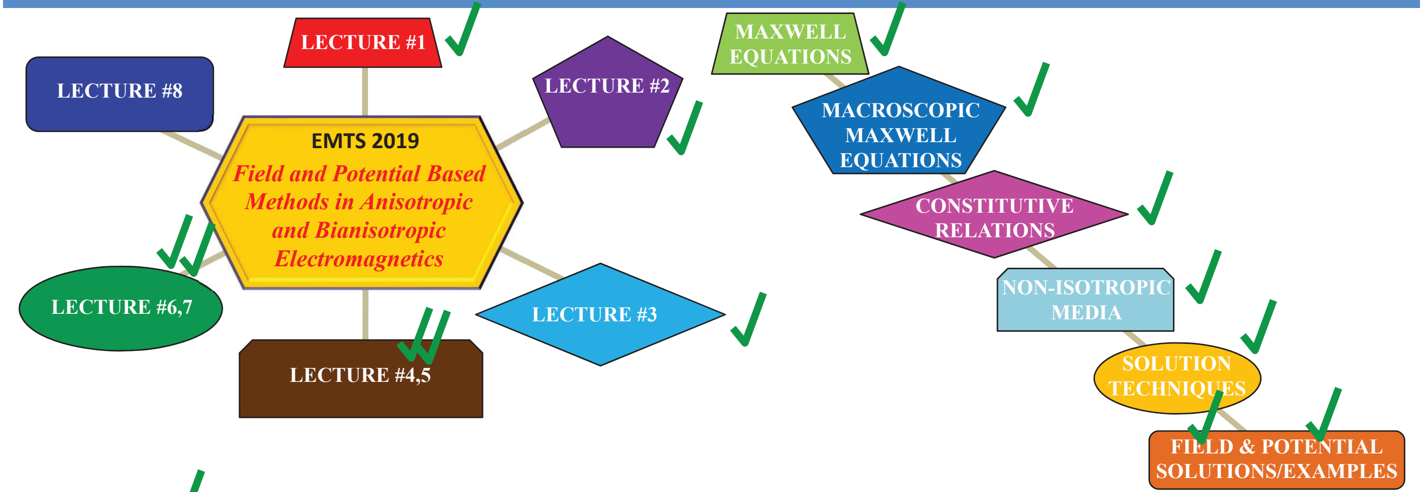
Overview – Lectures/Big Picture



LECTURE #1: Review various forms and regimes of validity of Maxwell's equations and constitutive relations.
LECTURE #2: Discuss factors that influence anisotropy and bianisotropy.
LECTURE #3: Discuss field and potential based methods for solving electromagnetic problems.
LECTURE #4,5: Provide several examples of field based solutions in complex media within and exterior to source regions.
LECTURE #6,7: Provide several examples of potential based solutions in complex media within and exterior to source regions.
LECTURE #8: Summary, conclusions and future research.

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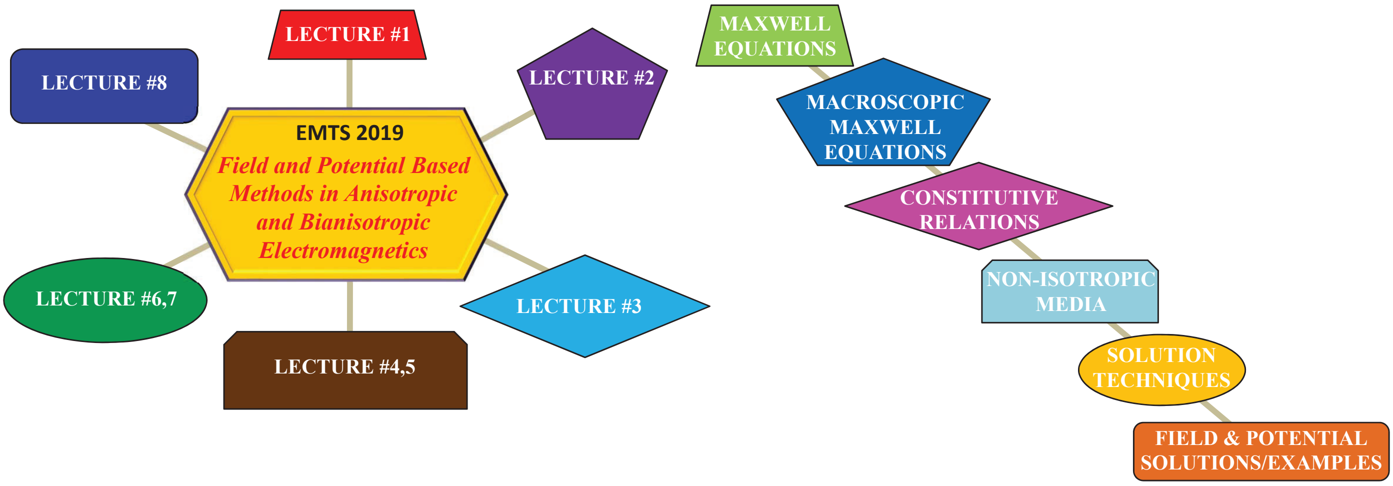
LECTURE #8
Conclusion

Dr. Michael J. Havrilla
Professor
Air Force Institute of Technology
WPAFB, Ohio 45433

Overview – Lectures/Big Picture



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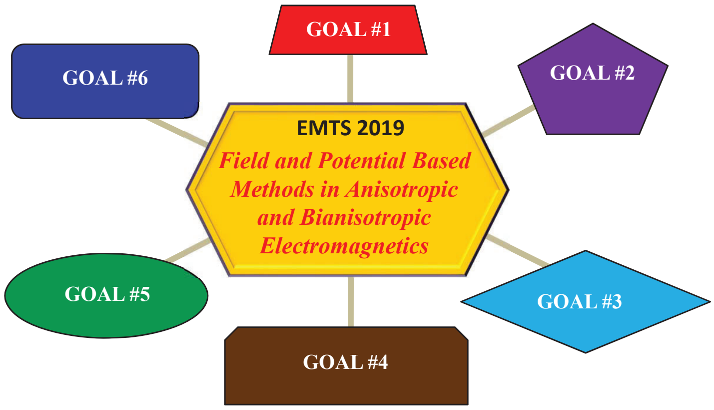
Overview – Primary Goals



GOAL #1: Gain a deeper appreciation of Maxwell's equations and the regimes of validity.
GOAL #2: Develop a better understanding of constitutive relations and recent areas of electromagnetic material research.
GOAL #3: Understand the profound influence that symmetry has on material tensor properties and design.
GOAL #4: Learn how to solve Maxwell's equations involving complex media using field and potential based techniques.
GOAL #5: Obtain deeper physical insight into electromagnetic field behavior in non-isotropic environments.
GOAL #6: Apply knowledge learned in your own personal research.

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Conferences

A SELECTION OF CONFERENCES DISCUSSING EM THEORY/MATERIALS

1. **EMTS.**
2. **Metamaterials.**
3. **Nanometa.**
4. **APS/URSI.**
5. **URSI General Assembly**
6. **ICEAA**

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Constitutive Relations – Outside the SIMPLE Media Box

Lots of research going on outside the SIMPLE media box!

COMPLEX MEDIA

Nonlinear, Bianisotropic, Spatially Inhomogeneous, Spatially Nonlocal, Time-Varying, Temporally Dispersive (plus other effects such as heat dependent, stress dependent, etc.)

SIMPLE MEDIA
Linear, Isotropic, Spatially Homogeneous
Spatially Local, Time Invariant, Temporally Dispersive

Where do you want to explore?!!!

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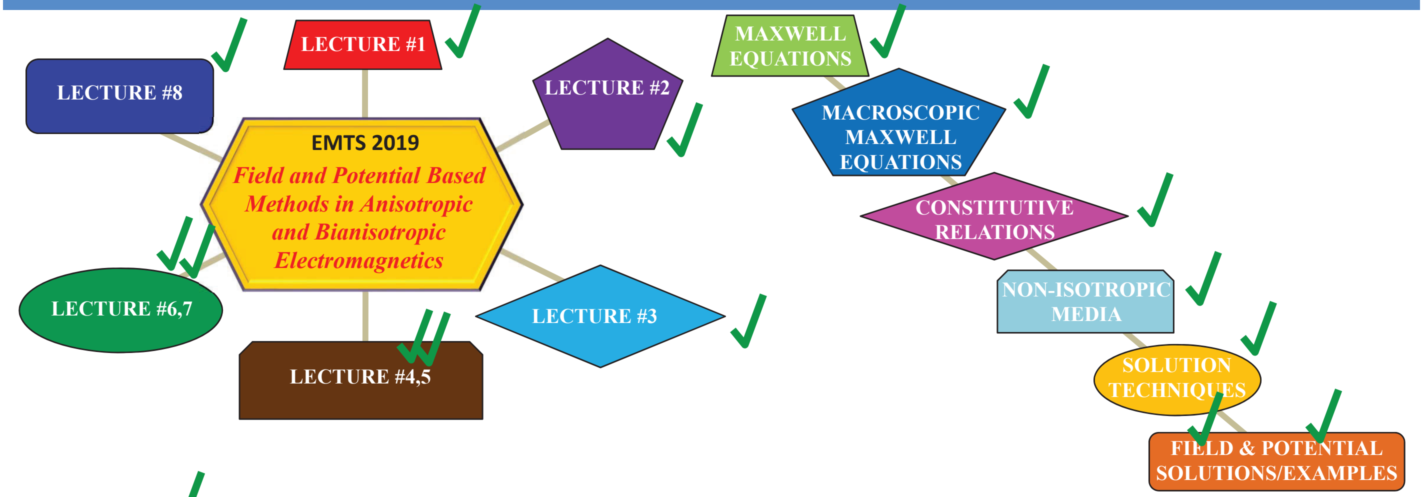
Overview – Lectures/Big Picture



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Fini – Thank you for attending!

Field and Potential Based Methods in Anisotropic and Bianisotropic Electromagnetics

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