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Electromagnetic fields and waves: mathematical models and numerical methods

Compendium

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The course focuses on foundations of the mathematical theory of electromagnetic fields and waves and includes: review of differential operations and theorems of the vector analysis; basic notions of the electromagnetic theory, Maxwell's equations; statements and analysis of the boundary value problems for Maxwell's and Helmholtz equations in unbounded domains associated with the wave diffraction, plane waves, conditions at infinity; statements and analysis of the boundary value problems for Maxwells and Helmholtz equations associated with the wave propagation in guides, the mathematical nature of electromagnetic waves; introduction to the integral equation method with application to the solution of the boundary value problems for the Helmholtz equation; and introduction to the theory of numerical solution of partial differential equations by finite difference, finite element, and Galerkin methods. The course offers a possibility of solving practical exercises and problems based on the considered basic theoretical items.

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1. Course description

The course focuses on foundations of the mathematical theory of electromagnetic fields and waves and includes:

- Basic electromagnetic theory. Differential operations of the vector analysis. Maxwell's and Helmholtz equations.
- Statements and analysis of the boundary value problems for Maxwell's and Helmholtz equations in unbounded domains associated with the wave diffraction. Conditions at infinity.
- Statements and analysis of the boundary value problems for Maxwell's and Helmholtz equations associated with the wave propagation in guides.
- Introduction to the integral equation method with application to the solution of the boundary value problems for the Helmholtz equation.
- Introduction to the theory of numerical solution of partial differential equations by finite difference, finite element, and Galerkin methods.

2. Introduction

Mathematical problems arising in electromagnetics and acoustics always attracted attention of mathematicians. The traditional (physical) diffraction theory was created by Huygens (who formulated in 1660s the famous Huygens principle, according to which the wave propagation is caused by secondary sources), Fresnel (1818), Maxwell (1850s), Helmholtz (1859), Kirchhof (1882), and others. The modern level was achieved owing to the studies of Poincare (1892) and Sommerfeld (1896) when it became clear that the mathematical theory of diffraction is connected with certain nonselfadjoint boundary value problems (BVPs) for partial differential equations (PDEs) of mathematical physics in unbounded domains. Recently, this theory has been developed on a completely new mathematical level involving the theory of distributions (Colton, Kress, Kleinmann, Werner, Vineberg, Costabel, Stephan, and others). The corresponding aspects of the theory of pseudodifferential operators were developed by Kohn and Nirenberg (1965), Eskin (1973), Shubin (1978), Taylor (1981), Rempel and Schulze (1982), and Mazja (1970-80s) who considered the problems on manifolds with sharp edges. The typical BVPs associated with the wave diffraction are stated in domains that have noncompact boundaries stretching to infinity and contain inclusions large in comparison with a characteristic parameter of the problem (e.g. wavelength) or dimensions of the boundary inhomogeneities. Recent remarkable progress of computational resources has opened new possibilities for solving such problems based on the use of huge computer clusters employing parallel computations.

These circumstances dictate the necessity of deeper studies of mathematical foundations of the electromagnetic field theory that would enable further development and creation of specifically oriented mathematical and numerical methods and techniques.

The course focuses on foundations of the mathematical theory of electromagnetic fields and waves and includes

(i) review of differential operations and theorems of the vector analysis;

(ii) basic notions of the electromagnetic theory, Maxwells equations;

(iii) statements and analysis of the boundary value problems (BVPs) for Maxwells and Helmholtz equations in unbounded domains associated with the wave diffraction, plane waves, conditions at infinity;

(iv) statements and analysis of the BVPs for Maxwells and Helmholtz equations associated with the wave propagation in guides, the mathematical nature of electromagnetic waves;

(v) introduction to the integral equation method with application to the solution of the BVPs for the Helmholtz equation; and

(vi) introduction to the theory of numerical solution of PDEs by finite difference, finite element, and Galerkin methods. The course offers a possibility of solving practical exercises and problems based on the considered basic theoretical items.

A compact version of the course presented at the URSI Commission B School for Young Scientists consists of three parts according to the following lists of topics .

Part 1

Differential operations and theorems of the vector analysis.

Basic electromagnetic theory. Maxwells and Helmholtz equations.

Statements and analysis of the boundary value problems (BVPs) for Maxwells and Helmholtz equations in unbounded domains associated with the wave diffraction. Plane waves. Conditions at infinity

Part 2

Statements and analysis of the BVPs for Maxwells and Helmholtz equations associated with the wave propagation in guides. The mathematical nature of waves.

Introduction to the integral equation (IE) method with application to the solution of the BVPs for the Helmholtz equation.

Introduction to the theory of numerical solution of ordinary and partial differential equations by finite difference, finite element, and Galerkin methods.

Part 3

A review of the statements and methods concerning basic proofs of the existence and uniqueness of the BVPs for Maxwells and Helmholtz in electromagnetic field theory.

Solution to some of the course problems.

The full version of the course consists of approximately 16 lectures presenting foundations of the mathematical theory of electromagnetic fields and waves.

A course plan. Full version

- Lecture 1: Differential operations of the vector analysis. Sections 8.1–8.7, 12.
- Lecture 2: Introduction to harmonic analysis. Section 3.
- Lecture 3: Basic electromagnetic theory. Maxwell’s and Helmholtz equations. Section 4.
- Lecture 4: Statements and analysis of the BVPs for Maxwell’s and Helmholtz equations in bounded domains. Integral equation method. Sections 5.1, 5.3, 5.4, 5.5, 5.7.
- Lecture 5: Statements and analysis of the BVPs for Maxwell’s and Helmholtz equations in unbounded domains associated with the wave diffraction. Conditions at infinity. Sections 5.2, 5.8, 5.9, 5.10.
- Lecture 6: Statements and analysis of the BVPs for Maxwell’s and Helmholtz equations associated with the wave propagation in guides. Sections 5.6, 7.1, 7.2, 7.3.
- Lecture 7: Introduction to the theory of numerical solution of PDEs by finite difference, finite element, and Galerkin methods. Section 16.
- Lecture 8: Repetition. Overview of the course problems and miniprojects.
- Additional parts of the course include the following more specialized topics:
  - numerical solution of BVPs for Maxwell’s equations in infinite domains, absorbing boundary conditions.

- a mixed Lee-Madsen finite element formulation for the numerical solution of Maxwell's equations in the time domain. Finite Difference Time-Domain method (Yee scheme) for the numerical solution of Maxwell's equations. Dispersion relation and stability analysis for the Yee scheme. Vector finite elements for solution of Maxwell's equations
- Interior Penalty Discontinuous Galerkin Finite Element Method (IPDGFEM) for the numerical solution of Maxwell's equations. A posteriori error analysis and adaptive error control.
- fully explicit hybrid IPDGFEM/FDM method for the numerical solution to Maxwell's equations.
- adaptive hybrid FEM/FDM method for the solution of inverse problems to Maxwell's equations.

Course problems.

There are about 25 problems in the course that accompany the theoretical lecture material. Those that are not marked with a star are on average level of difficulty, while miniprojects constitute more complicated problems.

It is assumed that a student would solve minimum 15 problems in written form and at least one miniproject; the latter may be solved by a small team consisting of two students.

Computations performed by standard accessible software (MATLAB) illustrating the solutions are encouraged, especially for the miniprojects.

Course literature

A.N. Tikhonov, A.A. Samarskij, *Equations of Mathematical Physics*, Dover Publications, 1990. ISBN 0-486-66422-8.

D. Colton and R. Kress, *Integral Equation Methods in Scattering Theory*, Wiley-Interscience Publication, New York (1983).

D. Colton and R. Kress, *Inverse Acoustic and Electromagnetic Scattering Theory*, Applied Mathematical Sciences 93, Springer-Verlag, Heidelberg (1992), 2nd Edition, 1998.

D. J. Jin, *The Finite Element Method in Electromagnetics,* Second Edition. J.Wiley and sons, 2002.

P.B.Monk, *Finite Element Methods for Maxwell's equations,* Oxford university press, 2003.

S. Larsson, V. Thomee, *Partial Differential Equations with Numerical Methods*, Applied Mathematical Sciences, Springer-Verlag, Heidelberg (2003).

A.Taflove, *Advances in computational electromagnetics: the finite difference time-domain method,* Boston, M-house, 1998.

Recommended reference literature

R. Adams, *Calculus,* 4th Edition. Addison-Wesley, 1999. ISBN 0-201-39607-6.

E. Kreyszig, *Advanced Engineering Mathematics,* 8th Edition. ISBN 0-471-33328-X.

In the compendium, the problem numbers, i.e. PROBLEM *a.b.*c corresponds the numbers in the book *E. Kreyszig, Advanced Engineering Mathematics, 8th Edition (AEM)*; for example, Problem *8.1.1* is problem 8.1.1 on p. 407 in *AEM*, chapter 8.1. The same applies to the numbers of theorems.

3. Harmonic functions

Various aspects of the mathematical theory of wave diffraction and propagation are connected with the analysis of the Helmholtz equation; its solutions are called meta-harmonic functions. Many important properties and statements of this theory have their roots in the theory of harmonic functions—solutions to the Laplace equation: statements of BVPs, potentials, fundamental solutions, integral equation method, proofs of existence and uniqueness etc. Therefore we begin the course with a basic introduction to the theory of harmonic functions.

3.1 Definition. Fundamental solutions

Denote by $D \in \mathrm{R}^2$ a two-dimensional domain bounded by the closed smooth contour Γ.

A twice continuously differentiable real-valued function u defined on a domain D is called *harmonic* if it satisfies Laplace's equation

$$\Delta u = 0 \quad \text{in} \quad D, \tag{3.1}$$

where

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \tag{3.2}$$

is called *Laplace operator* (Laplacian), the function $u = u(\mathbf{x})$, and $\mathbf{x} = (x, y) \in \mathrm{R}^2$. We will also use the notation $\mathbf{y} = (x_0, y_0)$.

The function

$$\Phi(\mathbf{x}, \mathbf{y}) = \Phi(\mathbf{x} - \mathbf{y}) = \frac{1}{2\pi} \ln \frac{1}{|\mathbf{x} - \mathbf{y}|} \tag{3.3}$$

is called *the fundamental solution of the Laplace equation*. For a fixed $\mathbf{y} \in \mathrm{R}^2$, $\mathbf{y} \neq \mathbf{x}$, the function $\Phi(\mathbf{x}, \mathbf{y})$ is harmonic, i.e., satisfies Laplace's equation

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0 \quad \text{in} \quad D. \tag{3.4}$$

The proof follows by straightforward differentiation.

3.2 Green's formulas

Let us formulate Green's theorems which leads to the second and third Green's formulas.

Let $D \in \mathrm{R}^2$ be a (two-dimensional) domain bounded by the closed smooth contour Γ and $\frac{\partial}{\partial n_y}$ denote the directional derivative in the direction of unit normal vector n_y to the boundary Γ directed into the exterior of Γ and corresponding to a point $\mathbf{y} \in \Gamma$. Then for every function u which is once continuously differentiable in the closed domain $\bar{D} = D + \Gamma$, $u \in C^1(\bar{D})$, and every function v which is twice continuously differentiable in $\bar{D}$, $v \in C^2(\bar{D})$, Green's first theorem (Green's first formula) is valid

$$\int\limits_D\int (u\Delta v + \operatorname{grad} u \cdot \operatorname{grad} v) d\mathbf{x} = \int\limits_\Gamma u \frac{\partial v}{\partial n_y} dl_y, \tag{3.5}$$

where $\cdot$ denotes the inner product of two vector-functions. For $u \in C^2(\bar{D})$ and $v \in C^2(\bar{D})$, Green's second theorem (Green's second formula) is valid

$$\int\limits_D\int (u\Delta v - v\Delta u) d\mathbf{x} = \int\limits_\Gamma \left(u \frac{\partial v}{\partial n_y} - v \frac{\partial u}{\partial n_y} \right) dl_y, \tag{3.6}$$

Proof. Apply Gauss theorem

$$\int\limits_D\int \operatorname{div} A d\mathbf{x} = \int\limits_\Gamma n_y \cdot A dl_y \tag{3.7}$$

to the vector function (vector field) $A = (A_1, A_2) \in C^1(\bar{D})$ (with the components A_1, A_2 being once continuously differentiable in the closed domain $\bar{D} = D + \Gamma$, $A_i \in C^1(\bar{D})$, $i = 1, 2$) defined by

$$A = u \cdot \operatorname{grad} v = \left[u\frac{\partial v}{\partial x}, u\frac{\partial v}{\partial y} \right],$$

the components are

$$A_1 = u\frac{\partial v}{\partial x}, \quad A_2 = u\frac{\partial v}{\partial y}.$$

We have

$$\operatorname{div} A = \operatorname{div}(u \operatorname{grad} v) = \frac{\partial}{\partial x}\left(u\frac{\partial v}{\partial x} \right) + \frac{\partial}{\partial y}\left(u\frac{\partial v}{\partial y} \right) =$$

$$\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + u\frac{\partial^2 v}{\partial x^2} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} + u\frac{\partial^2 v}{\partial y^2} = \operatorname{grad} u \cdot \operatorname{grad} v + u\Delta v.$$

On the other hand, according to the definition of the normal derivative,

$$n_y \cdot A = n_y \cdot (u \operatorname{grad} v) = u(n_y \cdot \operatorname{grad} v) = u\frac{\partial v}{\partial n_y},$$

so that, finally,

$$\int\limits_D\int \operatorname{div} A d\mathbf{x} = \int\limits_D\int (u\Delta v + \operatorname{grad} u \cdot \operatorname{grad} v) d\mathbf{x}, \quad \int\limits_\Gamma n_y \cdot A d l_y = \int\limits_\Gamma u\frac{\partial v}{\partial n_y} dl_y, \tag{3.8}$$

which proves Green's first formula. By interchanging u and v and subtracting we obtain the desired Green's second formula (3.6).

Let a twice continuously differentiable function $u \in C^2(\bar{D})$ be harmonic in the domain D. Then *Green's third theorem (Green's third formula)* is valid

$$u(\mathbf{x}) = \int\limits_\Gamma \left(\Phi(\mathbf{x}, \mathbf{y}) \frac{\partial u}{\partial n_y} - u(\mathbf{y}) \frac{\partial \Phi(\mathbf{x}, \mathbf{y})}{\partial n_y} \right) dl_y, \quad \mathbf{x} \in D. \tag{3.9}$$

Proof. For $\mathbf{x} \in D$ we choose a circle

$$\Omega(\mathbf{x}, r) = \{\mathbf{y} : |\mathbf{x} - \mathbf{y}| = \sqrt{(x - x_0)^2 + (y - y_0)^2} = r\}$$

of radius r (the boundary of a vicinity of a point $\mathbf{x} \in D$) such that $\Omega(\mathbf{x}, r) \in D$ and direct the unit normal n to $\Omega(\mathbf{x}, r)$ into the exterior of $\Omega(\mathbf{x}, r)$. Then we apply Green's second formula (3.6) to the harmonic function u and $\Phi(\mathbf{x}, \mathbf{y})$ (which is also a harmonic function) in the domain $\mathbf{y} \in D : |\mathbf{x} - \mathbf{y}| > r$ to obtain

$$0 = \int\int_{D \setminus \Omega(\mathbf{x},r)} (u\Delta\Phi(\mathbf{x}, \mathbf{y}) - \Phi(\mathbf{x}, \mathbf{y})\Delta u) d\mathbf{x} = \tag{3.10}$$

$$\int_{\Gamma \cup \Omega(\mathbf{x},r)} \left(u(\mathbf{y}) \frac{\partial \Phi(\mathbf{x}, \mathbf{y})}{\partial n_y} - \Phi(\mathbf{x}, \mathbf{y}) \frac{\partial u(\mathbf{y})}{\partial n_y} \right) dl_y.$$

Since on $\Omega(\mathbf{x}, r)$ we have

$$\text{grad}_{\,y} \Phi(\mathbf{x}, \mathbf{y}) = \frac{n_y}{2\pi r},$$

a straightforward calculation using the mean-value theorem and the fact that

$$\int_{\Gamma} \frac{\partial v}{\partial n_y} dl_y = 0 \tag{3.11}$$

for a harmonic in D function v shows that

$$\lim_{r \to 0} \int_{\Omega(\mathbf{x},r)} \left(u(\mathbf{y}) \frac{\partial \Phi(\mathbf{x}, \mathbf{y})}{\partial n_y} - \Phi(\mathbf{x}, \mathbf{y}) \frac{\partial u(\mathbf{y})}{\partial n_y} \right) dl_y = u(\mathbf{x}),$$

which yields (3.9).

3.3 BVPs for Laplace equation

Formulate *the interior Dirichlet problem*: find a function u that is harmonic in a domain D bounded by the closed smooth contour Γ, continuous in $\bar{D} = D \cup \Gamma$ and satisfies the Dirichlet boundary condition:

$$\Delta u = 0 \quad \text{in} \quad D, \tag{3.12}$$

$$u|_{\Gamma} = -f, \tag{3.13}$$

where f is a given continuous function.

Formulate *the interior Neumann problem*: find a function u that is harmonic in a domain D bounded by the closed smooth contour Γ, continuous in $\bar{D} = D \cup \Gamma$ and satisfies the Neumann boundary condition

$$\left. \frac{\partial u}{\partial n} \right|_{\Gamma} = -g, \tag{3.14}$$

where g is a given continuous function.

Theorem 1 The interior Dirichlet problem has at most one solution.

Proof. Assume that the interior Dirichlet problem has two solutions, u_1 and u_2. Their difference $u = u_1 - u_2$ is a harmonic function that is continuous up to the boundary and satisfies the homogeneous boundary condition $u = 0$ on the boundary Γ of D. Then from the maximum–minimum principle or its corollary we obtain $u = 0$ in D, which proves the theorem.

Theorem 2 Two solutions of the interior Neumann problem can differ only by a constant. The exterior Neumann problem has at most one solution.

Proof. We will prove the first statement of the theorem. Assume that the interior Neumann problem has two solutions, u_1 and u_2. Their difference $u = u_1 - u_2$ is a harmonic function that is continuous up to the boundary and satisfies the homogeneous boundary condition $\frac{\partial u}{\partial n_y} = 0$ on the boundary Γ of D. Suppose that u is not constant in D. Then there exists a closed ball B contained in D such that

$$\int\int_B |\mathrm{grad}\, u|^2 d\mathbf{x} > 0.$$

From Green's first formula applied to a pair of (harmonic) functions u, $v = u$ and the interior D_h of a surface $\Gamma_h = \{\mathbf{x} - hn_x : \mathbf{x} \in \Gamma\}$ parallel to the Ds boundary Γ with sufficiently small $h > 0$,

$$\int\int_{D_h} (u\Delta u + |\mathrm{grad}\, u|^2) d\mathbf{x} = \int_\Gamma u \frac{\partial u}{\partial n_y} dl_y,$$

we have first

$$\int\int_B |\mathrm{grad}\, u|^2 d\mathbf{x} \leq \int\int_{D_h} |\mathrm{grad}\, u|^2 d\mathbf{x},$$

(because $B \in D_h$); next we obtain

$$\int\int_{D_h} |\mathrm{grad}\, u|^2 d\mathbf{x} \leq \int_\Gamma u \frac{\partial u}{\partial n_y} dl_y = 0.$$

Passing to the limit $h \to 0$ we obtain a contradiction

$$\int\int_B |\mathrm{grad}\, u|^2 d\mathbf{x} \leq 0.$$

Hence, the difference $u = u_1 - u_2$ must be constant in D.

3.4 Potentials with logarithmic kernels

In the theory of BVPs, the integrals

$$u(\mathbf{x}) = \int_C E(\mathbf{x},\mathbf{y})\xi(\mathbf{y})dl_y, \quad v(\mathbf{x}) = \int_C \frac{\partial}{\partial \mathbf{n}_y} E(\mathbf{x},\mathbf{y})\eta(\mathbf{y})dl_y \tag{3.15}$$

are called *the potentials*. Here, $\mathbf{x} = (x,y)$, $\mathbf{y} = (x_0,y_0) \in \mathrm{R}^2$; $E(\mathbf{x},\mathbf{y})$ is the fundamental solution of a second-order elliptic differential operator;

$$\frac{\partial}{\partial \mathbf{n}_y} = \frac{\partial}{\partial n_y}$$

is the normal derivative at the point $\mathbf{y}$ of the closed piecewise smooth boundary C of a domain in R^2; and $\xi(\mathbf{y})$ and $\eta(\mathbf{y})$ are sufficiently smooth functions defined on C. In the case of Laplace operator Δu,

$$E(\mathbf{x},\mathbf{y}) = \Phi(\mathbf{x}-\mathbf{y}) = \frac{1}{2\pi} \ln \frac{1}{|\mathbf{x}-\mathbf{y}|}. \tag{3.16}$$

In the case of the Helmholtz operator $\mathscr{L}(k) = \Delta + k^2$, one can take $E(\mathbf{x},\mathbf{y})$ in the form

$$E(\mathbf{x},\mathbf{y}) = \mathscr{E}(\mathbf{x}-\mathbf{y}) = \frac{i}{4}H_0^{(1)}(k|\mathbf{x}-\mathbf{y}|) = \frac{1}{2\pi}\ln\frac{1}{|\mathbf{x}-\mathbf{y}|} + h(k|\mathbf{x}-\mathbf{y}|), \tag{3.17}$$

where $H_0^{(1)}(z) = -4i\Phi(z) + \tilde{h}(z)$ is the Hankel function of the first kind and zero order (one of the so-called cylindrical functions) and $\Phi(\mathbf{x}-\mathbf{y}) = \frac{1}{2\pi}\ln\frac{1}{|\mathbf{x}-\mathbf{y}|}$ is the kernel of the two-dimensional single layer potential; $\tilde{h}(z)$ and $h(z)$ are continuously differentiable and their second derivatives have a logarithmic singularity.

3.4.1 Properties of potentials

Theorem 3 Let $D \in \mathrm{R}^2$ be a domain bounded by the closed smooth contour Γ. Then the kernel of the double-layer potential

$$V(\mathbf{x},\mathbf{y}) = \frac{\partial\Phi(\mathbf{x},\mathbf{y})}{\partial n_y}, \quad \Phi(\mathbf{x},\mathbf{y}) = \frac{1}{2\pi}\ln\frac{1}{|\mathbf{x}-\mathbf{y}|}, \tag{3.18}$$

is a continuous function on Γ for $\mathbf{x}$, $\mathbf{y} \in \Gamma$.

Proof. Performing differentiation we obtain

$$V(\mathbf{x},\mathbf{y}) = \frac{1}{2\pi}\frac{\cos\theta_{\mathbf{x},\mathbf{y}}}{|\mathbf{x}-\mathbf{y}|}, \tag{3.19}$$

where $\theta_{\mathbf{x},\mathbf{y}}$ is the angle between the normal vector $\mathbf{n}_y$ at the integration point $\mathbf{y} = (x_0, y_0)$ and vector $\mathbf{x}-\mathbf{y}$. Choose the origin $O = (0,0)$ of (Cartesian) coordinates at the point $\mathbf{y}$ on curve Γ so that the Ox axis goes along the tangent and the Oy axis, along the normal to Γ at this point. Then one can write the equation of curve Γ in a sufficiently small vicinity of the point $\mathbf{y}$ in the form

$$y = y(x).$$

The assumption concerning the smoothness of Γ means that $y_0(x_0)$ is a differentiable function in a vicinity of the origin $O = (0,0)$, and one can write a segment of the Taylor series

$$y = y(0) + xy'(0) + \frac{x^2}{2}y''(\eta x) = \frac{1}{2}x^2 y''(\eta x) \quad (0 \le \eta < 1)$$

because $y(0) = y'(0) = 0$. Denoting $r = |\mathbf{x}-\mathbf{y}|$ and taking into account that $\mathbf{x} = (x,y)$ and the origin of coordinates is placed at the point $\mathbf{y} = (0,0)$, we obtain

$$r = \sqrt{(x-0)^2 + (y-0)^2} = \sqrt{x^2+y^2} = \sqrt{x^2 + \frac{1}{4}x^4(y''(\eta x))^2} = x\sqrt{1+\frac{1}{4}x^2(y''(\eta x))^2};$$

$$\cos\theta_{\mathbf{x},\mathbf{y}} = \frac{y}{r} = \frac{1}{2}\frac{xy''(\eta x)}{\sqrt{1+x^2(1/4)(y''(\eta x))^2}},$$

and

$$\frac{\cos\theta_{\mathbf{x},\mathbf{y}}}{r} = \frac{y}{r^2} = \frac{1}{2}\frac{y''(\eta x)}{1+x^2(1/4)(y''(\eta x))^2}.$$

The curvature K of a plane curve is given by

$$K = \frac{y''}{(1+(y')^2)^{3/2}},$$

which yields $y''(0) = K(\mathbf{y})$, and, finally,

$$\lim_{r\to 0}\frac{\cos\theta_{\mathbf{x},\mathbf{y}}}{r} = \frac{1}{2}K(\mathbf{y})$$

which proves the continuity of the kernel (3.18).

Statement 2 (Gauss formula) Let $D \in \mathrm{R}^2$ be a domain bounded by the closed smooth contour Γ. For the double-layer potential with a constant density

$$v^0(\mathbf{x}) = \int\limits_{\Gamma}\frac{\partial\Phi(\mathbf{x},\mathbf{y})}{\partial n_y}dl_y, \quad \Phi(\mathbf{x},\mathbf{y}) = \frac{1}{2\pi}\ln\frac{1}{|\mathbf{x}-\mathbf{y}|}, \tag{3.20}$$

where the (exterior) unit normal vector n of Γ is directed into the exterior domain $\mathrm{R}^2\setminus\bar{D}$, we have

$$\begin{aligned} v^0(\mathbf{x}) &= -1, \quad \mathbf{x}\in D, \\ v^0(\mathbf{x}) &= -\frac{1}{2}, \quad \mathbf{x}\in\Gamma, \\ v^0(\mathbf{x}) &= 0, \quad \mathbf{x}\in\mathrm{R}^2\setminus\bar{D}. \end{aligned} \tag{3.21}$$

Proof follows for $\mathbf{x}\in\mathrm{R}^2\setminus\bar{D}$ from the equality

$$\int\limits_{\Gamma}\frac{\partial v(\mathbf{y})}{\partial n_y}dl_y = 0 \tag{3.22}$$

applied to $v(\mathbf{y}) = \Phi(\mathbf{x},\mathbf{y})$. For $\mathbf{x}\in D$ it follows from Green's third formula

$$u(\mathbf{x}) = \int\limits_{\Gamma}\left(\Phi(\mathbf{x},\mathbf{y})\frac{\partial u}{\partial n_y} - u(\mathbf{y})\frac{\partial\Phi(\mathbf{x},\mathbf{y})}{\partial n_y}\right)dl_y, \quad \mathbf{x}\in \quad D, \tag{3.23}$$

applied to $u(\mathbf{y}) = 1$ in D.

Note also that if we set

$$v^0(\mathbf{x}') = -\frac{1}{2}, \quad \mathbf{x}'\in\Gamma,$$

we can also write (3.21) as

$$v^0_{\pm}(\mathbf{x}') = \lim_{h\to\pm 0} v(\mathbf{x}+hn_{x'}) = v^0(\mathbf{x}') \pm \frac{1}{2}, \quad \mathbf{x}'\in\Gamma. \tag{3.24}$$

Corollary. Let $D\in\mathrm{R}^2$ be a domain bounded by the closed smooth contour Γ. Introduce the single-layer potential with a constant density

$$u^0(\mathbf{x}) = \int\limits_{\Gamma}\Phi(\mathbf{x},\mathbf{y})dl_y, \quad \Phi(\mathbf{x},\mathbf{y}) = \frac{1}{2\pi}\ln\frac{1}{|\mathbf{x}-\mathbf{y}|}. \tag{3.25}$$

For the normal derivative of this single-layer potential

$$\frac{\partial u^0(\mathbf{x})}{\partial n_x} = \int\limits_{\Gamma}\frac{\partial\Phi(\mathbf{x},\mathbf{y})}{\partial n_x}dl_y, \tag{3.26}$$

where the (exterior) unit normal vector n_x of Γ is directed into the exterior domain $\mathrm{R}^2 \setminus \bar{D}$, we have

$$\begin{aligned} \frac{\partial u^0(\mathbf{x})}{\partial n_x} &= 1, \quad \mathbf{x} \in D, \\ \frac{\partial u^0(\mathbf{x})}{\partial n_x} &= \frac{1}{2}, \quad \mathbf{x} \in \Gamma, \\ \frac{\partial u^0(\mathbf{x})}{\partial n_x} &= 0, \quad \mathbf{x} \in \mathrm{R}^2 \setminus \bar{D}. \end{aligned} \tag{3.27}$$

Theorem 4 Let $D \in \mathrm{R}^2$ be a domain bounded by the closed smooth contour Γ. The double-layer potential

$$v(\mathbf{x}) = \int\limits_{\Gamma} \frac{\partial \Phi(\mathbf{x},\mathbf{y})}{\partial n_y} \varphi(\mathbf{y}) dl_y, \quad \Phi(\mathbf{x},\mathbf{y}) = \frac{1}{2\pi} \ln \frac{1}{|\mathbf{x}-\mathbf{y}|}, \tag{3.28}$$

with a continuous density φ can be continuously extended from D to $\bar{D}$ and from $\mathrm{R}^2 \setminus \bar{D}$ to $\mathrm{R}^2 \setminus D$ with the limiting values on Γ

$$v_{\pm}(\mathbf{x}') = \int\limits_{\Gamma} \frac{\partial \Phi(\mathbf{x}',\mathbf{y})}{\partial n_y} \varphi(\mathbf{y}) dl_y \pm \frac{1}{2} \varphi(\mathbf{x}'), \quad \mathbf{x}' \in \Gamma, \tag{3.29}$$

or

$$v_{\pm}(\mathbf{x}') = v(\mathbf{x}') \pm \frac{1}{2} \varphi(\mathbf{x}'), \quad \mathbf{x}' \in \Gamma, \tag{3.30}$$

where

$$v_{\pm}(\mathbf{x}') = \lim_{h \to \pm 0} v(\mathbf{x} + h n_{x'}). \tag{3.31}$$

Proof.
1. Introduce the function

$$I(\mathbf{x}) = v(\mathbf{x}) - v^0(\mathbf{x}) = \int\limits_{\Gamma} \frac{\partial \Phi(\mathbf{x},\mathbf{y})}{\partial n_y} (\varphi(\mathbf{y}) - \varphi_0) dl_y \tag{3.32}$$

where $\varphi_0 = \varphi(\mathbf{x}')$ and prove its continuity at the point $\mathbf{x}' \in \Gamma$. For every $\varepsilon > 0$ and every $\eta > 0$ there exists a vicinity $C_1 \subset \Gamma$ of the point $\mathbf{x}' \in \Gamma$ such that

$$|\varphi(\mathbf{x}) - \varphi(\mathbf{x}')| < \eta, \quad \mathbf{x}' \in C_1. \tag{3.33}$$

Set

$$I = I_1 + I_2 = \int\limits_{C_1} \cdots + \int\limits_{\Gamma \setminus C_1} \cdots.$$

Then

$$|I_1| \le \eta B_1,$$

where B_1 is a constant taken according to

$$\int\limits_{\Gamma} \left| \frac{\partial \Phi(\mathbf{x},\mathbf{y})}{\partial n_y} \right| dl_y \le B_1 \quad \forall \mathbf{x} \in \overline{D}, \tag{3.34}$$

and condition (3.34) holds because kernel (3.18) is continuous on Γ.

Taking $\eta = \varepsilon/B_1$, we see that for every $\varepsilon > 0$ there exists a vicinity $C_1 \subset \Gamma$ of the point $\mathbf{x}' \in \Gamma$ (i.e., $\mathbf{x}' \in C_1$) such that

$$|I_1(\mathbf{x})| < \varepsilon \quad \forall \mathbf{x} \in \overline{D}. \tag{3.35}$$

Inequality (3.35) means that $I(\mathbf{x})$ is continuous at the point $\mathbf{x}' \in \Gamma$.
2. Now, if we take $v_\pm(\mathbf{x}')$ for $\mathbf{x}' \in \Gamma$ and consider the limits $v_\pm(\mathbf{x}') = \lim_{h\to\pm 0} v(\mathbf{x}+hn_y)$ from inside and outside contour Γ using (3.24), we obtain

$$v_-(\mathbf{x}') = I(\mathbf{x}') + \lim_{h\to -0} v^0(\mathbf{x}' - hn_{x'}) = v(\mathbf{x}') - \frac{1}{2}\varphi(\mathbf{x}'),$$

$$v_+(\mathbf{x}') = I(\mathbf{x}') + \lim_{h\to +0} v^0(\mathbf{x}' + hn_{x'}) = v(\mathbf{x}') + \frac{1}{2}\varphi(\mathbf{x}'),$$

which prove the theorem.

Corollary. Let $D \in \mathrm{R}^2$ be a domain bounded by the closed smooth contour Γ. Introduce the single-layer potential

$$u(\mathbf{x}) = \int_\Gamma \Phi(\mathbf{x},\mathbf{y})\varphi(\mathbf{y})dl_y, \quad \Phi(\mathbf{x},\mathbf{y}) = \frac{1}{2\pi}\ln\frac{1}{|\mathbf{x}-\mathbf{y}|}. \tag{3.36}$$

with a continuous density φ. The normal derivative of this single-layer potential

$$\frac{\partial u(\mathbf{x})}{\partial n_x} = \int_\Gamma \frac{\partial \Phi(\mathbf{x},\mathbf{y})}{\partial n_x}\varphi(\mathbf{y})dl_y \tag{3.37}$$

can be continuously extended from D to $\bar{D}$ and from $\mathrm{R}^2 \setminus \bar{D}$ to $\mathrm{R}^2 \setminus D$ with the limiting values on Γ

$$\left.\frac{\partial u(\mathbf{x}')}{\partial n_x}\right|_\pm = \int_\Gamma \frac{\partial \Phi(\mathbf{x}',\mathbf{y})}{\partial n_{x'}}\varphi(\mathbf{y})dl_y \mp \frac{1}{2}\varphi(\mathbf{x}'), \quad \mathbf{x}' \in \Gamma, \tag{3.38}$$

or

$$\left.\frac{\partial u(\mathbf{x}')}{\partial n_x}\right|_\pm = \frac{\partial u(\mathbf{x}')}{\partial n_x} \mp \frac{1}{2}\varphi(\mathbf{x}'), \quad \mathbf{x}' \in \Gamma, \tag{3.39}$$

where

$$\frac{\partial u(\mathbf{x}')}{\partial n_{x'}} = \lim_{h\to\pm 0} n_{x'} \cdot \operatorname{grad} v(\mathbf{x}' + hn_{x'}). \tag{3.40}$$

3.4.2 Generalized potentials

Let $S_\Pi(\Gamma) \in \mathrm{R}^2$ be a domain bounded by the closed piecewise smooth contour Γ. We assume that a rectilinear interval Γ_0 is a subset of Γ, so that $\Gamma_0 = \{\mathbf{x} : y = 0, x \in [a,b]\}$.

Let us say that functions $U_l(\mathbf{x})$ are *the generalized single layer (SLP)* $(l = 1)$ *or double layer (DLP)* $(l = 2)$ *potentials* if

$$U_l(\mathbf{x}) = \int_\Gamma K_l(\mathbf{x},t)\varphi_l(t)dt, \quad \mathbf{x} = (x,y) \in S_\Pi(\Gamma),$$

where

$$K_l(\mathbf{x},t) = g_l(\mathbf{x},t) + F_l(\mathbf{x},t) \quad (l = 1,2),$$

$$g_1(\mathbf{x},t) = g(x,y^0) = \frac{1}{\pi}\ln\frac{1}{|\mathbf{x}-\mathbf{y}^0|}, \quad g_2(\mathbf{x},t) = \frac{\partial}{\partial y_0}g(\mathbf{x},\mathbf{y}^0) \quad [\mathbf{y}^0 = (t,0)],$$

$F_{1,2}$ are smooth functions, and we shall assume that for every closed domain $S_{0\Pi}(\Gamma) \subset S_{\Pi}(\Gamma)$, the following conditions hold

i) $F_1(\mathbf{x},t)$ is once continuously differentiable with respect to the variables of $\mathbf{x}$ and continuous in t;
ii) $F_2(\mathbf{x},t)$ and

$$F_2^1(\mathbf{x},t) = \frac{\partial}{\partial y}\int_q^t F_2(x,s)ds, \quad q \in \mathrm{R}^1,$$

are continuous.

We shall also assume that the densities of the generalized potentials $\varphi_1 \in L_2^{(1)}(\Gamma)$ and $\varphi_2 \in L_2^{(2)}(\Gamma)$.

Generalized potentials keep all essential properties of harmonic potentials with a purely logarithmic kernel stated in the previous section.

3.5 Reduction of BVPs to integral equations

Green's formulas show that each harmonic function can be represented as a combination of single- and double-layer potentials. For BVPs we will find a solution in the form of one of these potentials.

Introduce integral operators K_0 and K_1 acting in the space $C(\Gamma)$ of continuous functions defined on contour Γ

$$K_0(\mathbf{x}) = 2\int_\Gamma \frac{\partial\Phi(\mathbf{x},\mathbf{y})}{\partial n_y}\varphi(\mathbf{y})dl_y, \quad \mathbf{x}\in\Gamma \tag{3.41}$$

and

$$K_1(\mathbf{x}) = 2\int_\Gamma \frac{\partial\Phi(\mathbf{x},\mathbf{y})}{\partial n_x}\psi(\mathbf{y})dl_y, \quad \mathbf{x}\in\Gamma. \tag{3.42}$$

The kernels of operators K_0 and K_1 are continuous on Γ As seen by interchanging the order of integration, K_0 and K_1 are adjoint as integral operators in the space $C(\Gamma)$ of continuous functions defined on curve Γ.

Theorem 5 The operators $I - K_0$ and $I - K_1$ have trivial nullspaces

$$N(I-K_0) = \{0\}, \quad N(I-K_1) = \{0\},$$

The nullspaces of operators $I + K_0$ and $I + K_1$ have dimension one and

$$N(I+K_0) = \operatorname{span}\{1\}, \quad N(I+K_1) = \operatorname{span}\{\psi_0\}$$

with

$$\int_\Gamma \psi_0 dl_y \neq 0.$$

Proof. Let φ be a solution to the homogeneous integral equation $\phi - K_0\phi = 0$ and define a double-layer potential

$$v(\mathbf{x}) = \int_\Gamma \frac{\partial\Phi(\mathbf{x},\mathbf{y})}{\partial n_y}\varphi(\mathbf{y})dl_y, \quad \mathbf{x}\in D \tag{3.43}$$

with a continuous density ϕ. Then we have, for the limiting values on Γ,

$$v_\pm(\mathbf{x}) = \int_\Gamma \frac{\partial\Phi(\mathbf{x},\mathbf{y})}{\partial n_y}\varphi(\mathbf{y})dl_y \pm \frac{1}{2}\varphi(\mathbf{x}) \tag{3.44}$$

which yields

$$2v_-(\mathbf{x}) = K_0\varphi(\mathbf{x}) - \varphi = 0. \tag{3.45}$$

From the uniqueness of the interior Dirichlet problem (Theorem 1) it follows that $v = 0$ in the whole domain D. Now we will apply the continuity of the normal derivative of the double-layer potential (3.43) over a smooth contour Γ,

$$\left.\frac{\partial v(\mathbf{y})}{\partial n_y}\right|_+ - \left.\frac{\partial v(\mathbf{y})}{\partial n_y}\right|_- = 0, \quad \mathbf{y} \in \Gamma, \tag{3.46}$$

where the internal and external limiting values with respect to Γ are defined as follows

$$\left.\frac{\partial v(\mathbf{y})}{\partial n_y}\right|_- = \lim_{\mathbf{y}\to\Gamma, \mathbf{y}\in D} \frac{\partial v(\mathbf{y})}{\partial n_y} = \lim_{h\to 0} n_y \cdot \operatorname{grad} v(\mathbf{y} - hn_y) \tag{3.47}$$

and

$$\left.\frac{\partial v(\mathbf{y})}{\partial n_y}\right|_+ = \lim_{\mathbf{y}\to\Gamma, \mathbf{y}\in \mathbf{R}^2\setminus\bar{D}} \frac{\partial v(\mathbf{y})}{\partial n_y} = \lim_{h\to 0} n_y \cdot \operatorname{grad} v(\mathbf{y} + hn_y). \tag{3.48}$$

Note that (3.46) can be written as

$$\lim_{h\to 0} n_y \cdot [\operatorname{grad} v(\mathbf{y} + hn_y) - \operatorname{grad} v(\mathbf{y} - hn_y)] = 0. \tag{3.49}$$

From (3.47)–(3.49) it follows that

$$\left.\frac{\partial v(\mathbf{y})}{\partial n_y}\right|_+ - \left.\frac{\partial v(\mathbf{y})}{\partial n_y}\right|_- = 0, \quad \mathbf{y} \in \Gamma. \tag{3.50}$$

Using the uniqueness for the exterior Neumann problem and the fact that $v(\mathbf{y}) \to 0$, $|\mathbf{y}| \to \infty$, we find that $v(\mathbf{y}) = 0$, $\mathbf{y} \in \mathrm{R}^2 \setminus \bar{D}$. Hence from (3.44) we deduce $\varphi = v_+ - v_- = 0$ on Γ. Thus $N(I - K_0) = \{0\}$ and, by the Fredholm alternative, $N(I - K_1) = \{0\}$.

Theorem 6 Let $D \in \mathrm{R}^2$ be a domain bounded by the closed smooth contour Γ. The double-layer potential

$$v(\mathbf{x}) = \int_\Gamma \frac{\partial \Phi(\mathbf{x},\mathbf{y})}{\partial n_y} \varphi(\mathbf{y}) dl_y, \quad \Phi(\mathbf{x},\mathbf{y}) = \frac{1}{2\pi} \ln \frac{1}{|\mathbf{x}-\mathbf{y}|}, \quad \mathbf{x} \in D, \tag{3.51}$$

with a continuous density φ is a solution of the interior Dirichlet problem provided that φ is a solution of the integral equation

$$\varphi(\mathbf{x}) - 2\int_\Gamma \frac{\partial \Phi(\mathbf{x},\mathbf{y})}{\partial n_y} \varphi(\mathbf{y}) dl_y = -2f(x), \quad \mathbf{x} \in \Gamma, \tag{3.52}$$

where $f(x)$ is given by (3.13).

Proof follows from Theorem 4.

Theorem 7 The interior Dirichlet problem has a unique solution.

Proof The integral equation $\phi - K\phi = -2f$ of the interior Dirichlet problem has a unique solution by Theorem 5 because $N(I - K) = \{0\}$.

Theorem 8 Let $D \in \mathrm{R}^2$ be a domain bounded by the closed smooth contour Γ. The double-layer potential

$$u(\mathbf{x}) = \int\limits_{\Gamma} \frac{\partial \Phi(\mathbf{x},\mathbf{y})}{\partial n_y} \varphi(\mathbf{y}) dl_y, \quad \mathbf{x} \in \mathrm{R}^2 \setminus \bar{D}, \tag{3.53}$$

with a continuous density φ is a solution of the exterior Dirichlet problem provided that φ is a solution of the integral equation

$$\varphi(\mathbf{x}) + 2\int\limits_{\Gamma} \frac{\partial \Phi(\mathbf{x},\mathbf{y})}{\partial n_y} \varphi(\mathbf{y}) dl_y = 2f(x), \quad \mathbf{x} \in \Gamma. \tag{3.54}$$

Here we assume that the origin is contained in D.

Proof follows from Theorem 4.

Theorem 9 The exterior Dirichlet problem has a unique solution.

Proof. The integral operator $\tilde{K} : C(\Gamma) \to C(\Gamma)$ defined by the right-hand side of (3.53) is compact in the space $C(\Gamma)$ of functions continuous on Γ because its kernel is a continuous function on Γ. Let φ be a solution to the homogeneous integral equation $\varphi + \tilde{K}\varphi = 0$ on Γ and define u by (3.53). Then $2u = \varphi + \tilde{K}\varphi = 0$ on Γ, and by the uniqueness for the exterior Dirichlet problem it follows that $u \in \mathrm{R}^2 \setminus \bar{D}$, and $\int_\Gamma \varphi dl_y = 0$. Therefore $\varphi + K\varphi = 0$ which means according to Theorem 5 (because $N(I+K) = \mathrm{span}\,\{1\}$), that $\varphi = const$ on Γ. Now $\int_\Gamma \varphi dl_y = 0$ implies that $\varphi = 0$ on Γ, and the existence of a unique solution to the integral equation (3.54) follows from the Fredholm property of its operator.

Theorem 10 Let $D \in \mathrm{R}^2$ be a domain bounded by the closed smooth contour Γ. The single-layer potential

$$u(\mathbf{x}) = \int\limits_{\Gamma} \Phi(\mathbf{x},\mathbf{y}) \psi(\mathbf{y}) dl_y, \quad \mathbf{x} \in D, \tag{3.55}$$

with a continuous density ψ is a solution of the interior Neumann problem provided that ψ is a solution of the integral equation

$$\psi(\mathbf{x}) + 2\int\limits_{\Gamma} \frac{\partial \Phi(\mathbf{x},\mathbf{y})}{\partial n_x} \psi(\mathbf{y}) dl_y = 2g(x), \quad \mathbf{x} \in \Gamma. \tag{3.56}$$

Theorem 11 The interior Neumann problem is solvable if and only if

$$\int_\Gamma \psi dl_y = 0 \tag{3.57}$$

is satisfied.

3.6 Problems

3.6.1 Problem

Prove that the function (3.3)

$$\Phi(\mathbf{x},\mathbf{y}) = \Phi(\mathbf{x}-\mathbf{y}) = \frac{1}{2\pi} \ln \frac{1}{|\mathbf{x}-\mathbf{y}|}$$

(the fundamental solution of the Laplace equation) is harmonic with respect to the coordinates of $\mathbf{x}$ for a fixed $\mathbf{y} \in \mathrm{R}^2$, $\mathbf{y} \neq \mathbf{x}$ and with respect to $\mathbf{y}$ for a fixed $\mathbf{x} \in \mathrm{R}^2$, $\mathbf{x} \neq \mathbf{y}$.

3.6.2 Problem

Prove that the function (3.17)

$$E(\mathbf{x},\mathbf{y}) = \mathscr{E}(\mathbf{x}-\mathbf{y}) = \frac{i}{4}H_0^{(1)}(k|\mathbf{x}-\mathbf{y}|) = \frac{1}{2\pi}\ln\frac{1}{|\mathbf{x}-\mathbf{y}|} + h(k|\mathbf{x}-\mathbf{y}|),$$

is the kernel of a generalized potential according to the definition of Section 3.4.2.

3.6.3 Problem

Reduce to a boundary integral equation the BVP from Section 14.2 in a rectangle $\Pi_{ab} = \{(x,y) : 0 < x < a, 0 < y < b\}$

$$\begin{cases} \Delta u = 0, \quad u = u(x,y), \ \ 0 < x < a, \ 0 < y < b, \qquad u \in C^2(\Pi_{ab}) \cap C(\bar{\Pi}_{ab}) \\ u(0,y) = 0, \qquad u(a,y) = 0, \ \ 0 \le y \le b, \\ u(x,0) = 0, \qquad u(x,b) = H(x), \ \ 0 \le x \le a, \end{cases}$$

$$H(x) = \begin{cases} Q[p^2 - (x-x_S)^2]^2 e^{-r(x-x_S)^2}, \ |x-x_S| \le p, \\ 0, \quad |x-x_S| \ge p, \end{cases} \tag{3.58}$$

with $supp H(x) = L = (x_S - p, x_S + p) \subset (0,a)$.

3.6.4 Problem

Formulate a Neumann BVP in a rectangle Π_{ab} as in 3.6.3 with a compactly supported boundary function of your own. Check the solvability condition (3.57).

4. Maxwell's equations

The classical macroscopic electromagnetic field is described by four three-component vector-functions $\mathbf{E}(\mathbf{r},t)$, $\mathbf{D}(\mathbf{r},t)$, $\mathbf{H}(\mathbf{r},t)$, and $\mathbf{B}(\mathbf{r},t)$ of the position vector $\mathbf{r} = (x,y,z)$ and time t. The fundamental field vectors $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{H}(\mathbf{r},t)$ are called *electric* and *magnetic field intensities*. $\mathbf{D}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$ which will be eliminated from the description via constitutive relations are called *the electric displacement* and *magnetic induction*. The fields and sources are related by *the Maxwell equation system*

$$\frac{\partial \mathbf{D}}{\partial t} - \mathrm{rot}\mathbf{H} = -\mathbf{J}, \tag{4.1}$$

$$\frac{\partial \mathbf{B}}{\partial t} + \mathrm{rot}\mathbf{E} = 0, \tag{4.2}$$

$$\mathrm{div}\,\mathbf{B} = 0, \tag{4.3}$$

$$\mathrm{div}\,\mathbf{D} = \rho, \tag{4.4}$$

written in the standard SI units.

The constitutive relations are

$$\mathbf{D} = \varepsilon\mathbf{E}, \tag{4.5}$$

$$\mathbf{B} = \mu\mathbf{H}, \tag{4.6}$$

$$\mathbf{J} = \sigma\mathbf{E}. \tag{4.7}$$

Here ε, μ, and σ, which are generally bounded functions of position (the first two are assumed positive), are permittivity, permeability, and conductivity of the medium for $\mathbf{J}$ being the conductivity current density.

In vacuum, that is, in a homogeneous medium with constant characteristics $\varepsilon = \varepsilon_0$, $\mu = \mu_0$, and

$\sigma = 0$, the Maxwell equation system takes a simpler form

$$\begin{aligned} \operatorname{rot}\mathbf{H} &= \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}, && (4.8)\\ \operatorname{rot}\mathbf{E} &= -\mu_0 \frac{\partial \mathbf{H}}{\partial t}, && (4.9)\\ \operatorname{div}\mathbf{H} &= 0, && (4.10)\\ \operatorname{div}\mathbf{E} &= \rho. && (4.11) \end{aligned}$$

Note that divergence conditions (4.3), (4.4), (4.10), and (4.11) follows from fundamental field equations (4.1), (4.2), (4.8), and (4.9). Indeed, taking the divergence of (4.1) and (4.2) and recalling that $\operatorname{div}\operatorname{rot}\mathbf{A} = 0$ for any vector-function $\mathbf{A}$, we obtain

$$\operatorname{div}\frac{\partial \mathbf{B}}{\partial t} = 0, \quad \operatorname{div}\frac{\partial \mathbf{D}}{\partial t} = -\operatorname{div}\mathbf{J}.$$

Since charge is conserved, the charge function ρ and conductivity current $\mathbf{J}$ are coupled by the relation

$$\operatorname{div}\mathbf{J} + \frac{\partial \rho}{\partial t} = 0; \tag{4.12}$$

hence,

$$\frac{\partial}{\partial t}\operatorname{div}\mathbf{B} = \frac{\partial}{\partial t}(\operatorname{div}\mathbf{D} - \rho) = 0.$$

However, the divergence conditions should be taken into account when designing a numerical scheme on the stage of discretization.

In the case of a homogeneous medium, it is reasonable to obtain equations for each vector $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{H}(\mathbf{r},t)$. To this end, assume that $\rho = 0$. Applying the operation rot to equation (4.1) and taking into account the constitutive relations, we have

$$\operatorname{rot}\operatorname{rot}\mathbf{H} = \varepsilon \frac{\partial}{\partial t}\operatorname{rot}\mathbf{E} + \sigma \operatorname{rot}\mathbf{E}. \tag{4.13}$$

Using the vector differential identity $\operatorname{rot}\operatorname{rot}\mathbf{A} = \operatorname{grad}\operatorname{div}\mathbf{A} - \Delta\mathbf{A}$ and taking into notice equation (4.2), we obtain the equation for magnetic field $\mathbf{H}$

$$\operatorname{grad}\operatorname{div}\mathbf{H} - \Delta\mathbf{H} = -\varepsilon\mu \frac{\partial^2 \mathbf{H}}{\partial t^2} - \sigma\mu \frac{\partial \mathbf{H}}{\partial t}$$

or

$$\Delta\mathbf{H} = \frac{1}{a^2}\frac{\partial^2 \mathbf{H}}{\partial t^2} + \sigma\mu \frac{\partial \mathbf{H}}{\partial t} \qquad \left(a^2 = \frac{1}{\varepsilon\mu}\right) \tag{4.14}$$

because $\operatorname{div}\mathbf{H} = 0$.

The same equation holds for electric field $\mathbf{E}$

$$\Delta\mathbf{E} = \frac{1}{a^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} + \sigma\mu \frac{\partial \mathbf{E}}{\partial t} \qquad \left(a^2 = \frac{1}{\varepsilon\mu}\right). \tag{4.15}$$

Equations (4.14) or (4.15) hold for all field components,

$$\Delta u = \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} + \sigma\mu \frac{\partial u}{\partial t}, \tag{4.16}$$

where u is one of the components H_x, H_y, H_z or E_x, E_y, E_z.

If the medium is nonconducting, $\sigma = 0$, then (4.14), (4.15), or (4.16) yield a standard wave equation

$$\Delta u = \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2}. \tag{4.17}$$

This implies that electromagnetic processes are actually *waves* that propagate in the medium with the speed $a = \dfrac{1}{\sqrt{\varepsilon\mu}}$ (the latter holds for vacuum).

4.1 Time-harmonic fields

Time-periodic (time-harmonic) fields

$$\mathbf{H}(\mathbf{r},t) = \mathbf{H}(\mathbf{r})e^{-i\omega t}, \qquad \mathbf{E}(\mathbf{r},t) = \mathbf{E}(\mathbf{r})e^{-i\omega t} \tag{4.18}$$

constitute a very important particular case. Functions $\mathbf{E}$ and $\mathbf{H}$ are the field complex amplitudes; the quantities $\mathrm{Re}\,\mathbf{E}$ and $\mathrm{Re}\,\mathbf{H}$ have direct physical meaning.

Assuming that complex electromagnetic field (4.18) satisfies Maxwell equations and that the currents are also time-harmonic, $\mathbf{J}(\mathbf{r},t) = \mathbf{J}(\mathbf{r})e^{-i\omega t}$, substitute (4.18) into (4.1)–(4.4) to obtain

$$\mathrm{rot}\mathbf{H} = -i\omega\mathbf{D} + \mathbf{J}, \tag{4.19}$$
$$\mathrm{rot}\mathbf{E} = i\omega\mathbf{B}, \tag{4.20}$$
$$\mathrm{div}\,\mathbf{B} = 0, \tag{4.21}$$
$$\mathrm{div}\,\mathbf{D} = \rho. \tag{4.22}$$

Since $\mathbf{J} = \sigma\mathbf{E}$, equation (4.19) can be transformed by introducing the complex permittivity

$$\varepsilon' = \varepsilon + i\frac{\sigma}{\omega}.$$

As a result, system (4.19)–(4.22) takes the form

$$\mathrm{rot}\,\mathbf{H} = -i\omega\varepsilon'\mathbf{E}, \tag{4.23}$$
$$\mathrm{rot}\,\mathbf{E} = i\omega\mu\mathbf{H}, \tag{4.24}$$
$$\mathrm{div}\,(\mu\mathbf{H}) = 0, \tag{4.25}$$
$$\mathrm{div}\,(\varepsilon\mathbf{E}) = \rho. \tag{4.26}$$

In a homogeneous medium and when external currents are absent, equations (4.25) and (4.26) follow from the first two Maxwell equations (4.23) and (4.24).

4.2 Simplest solutions: plane waves

Consider the simplest time-harmonic solutions to Maxwell equations in a homogeneous medium (with constant characteristics), plane electromagnetic waves. In the absence of free charges when $\mathrm{div}\,\mathbf{E} = 0$, the electric field vector satisfies the equation

$$\mathrm{rotrot}\mathbf{E} = \omega^2\varepsilon'\mu\mathbf{E}, \tag{4.27}$$

or

$$\Delta\mathbf{E} + \kappa^2\mathbf{E} = 0, \tag{4.28}$$

where

$$\varepsilon' = \varepsilon + i\frac{\sigma}{\omega}, \quad \kappa^2 = \omega^2 \varepsilon' \mu = k^2 + i\omega\mu\sigma, \quad k = \omega\sqrt{\varepsilon\mu}. \tag{4.29}$$

In the cartesian coordinate system, equation (4.28) holds for every field component,

$$\Delta u + \kappa^2 u = 0, \tag{4.30}$$

where u is one of the components E_x, E_y, E_z.

The Helmholtz equation (4.30) has a solution in the form of a plane wave; componentwise,

$$E_\alpha = E_\alpha^0 e^{i(\kappa_x x + \kappa_y y + \kappa_z z)}, \quad \kappa_x^2 + \kappa_y^2 + \kappa_z^2 = \kappa^2 \quad (\alpha = x, y, z). \tag{4.31}$$

Here κ is called the wave propagation constant. Therefore, the vector Helmholtz equation (4.28) has a solution

$$\mathbf{E} = \mathbf{E}_0 e^{i(\kappa_x x + \kappa_y y + \kappa_z z)} = \mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}}, \tag{4.32}$$

where the vectors

$$\mathbf{k} = (\kappa_x, \kappa_y, \kappa_z), \quad \mathbf{r} = (x, y, z), \quad \mathbf{E}_0 = const. \tag{4.33}$$

Since $\operatorname{div}\mathbf{E} = 0$, we have

$$\operatorname{div}\mathbf{E} = \operatorname{div}(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}}) = i e^{i\mathbf{k}\cdot\mathbf{r}}\mathbf{k}\cdot\mathbf{E}_0 = 0.$$

Thus, $\mathbf{k}\cdot\mathbf{E}_0 = 0$ so that the direction of vector $\mathbf{E}$ is orthogonal to the direction of the plane wave propagation governed by vector $\mathbf{k}$.

Vectors $\mathbf{E}$ and $\mathbf{H}$ are coupled by the relation

$$\operatorname{rot}\mathbf{E} = i\omega\mu\mathbf{H}. \tag{4.34}$$

Since

$$\operatorname{rot}(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}}) = [\operatorname{grad} e^{i\mathbf{k}\cdot\mathbf{r}}, \mathbf{E}_0],$$

we have

$$\sqrt{\varepsilon'}[\mathbf{k}_0, \mathbf{E}_0] = \sqrt{\mu}\mathbf{H}_0, \tag{4.35}$$

where $\mathbf{k}_0 = \mathbf{k}/|\mathbf{k}|$ is the unit vector in the direction of the wave propagation. Thus, vectors $\mathbf{E}$ and $\mathbf{H}$ are not only orthogonal to the direction of the wave propagation but also mutually orthogonal:

$$\mathbf{E}\cdot\mathbf{H} = 0, \quad \mathbf{E}\cdot\mathbf{k} = 0, \quad \mathbf{H}\cdot\mathbf{k} = 0. \tag{4.36}$$

We see that the Maxwell equations have a solution in the form of a plane electromagnetic wave

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}}, \quad \mathbf{H}(\mathbf{r}) = \mathbf{H}_0 e^{i\mathbf{k}\cdot\mathbf{r}}, \tag{4.37}$$

where

$$\sqrt{\varepsilon'}[\mathbf{k}_0, \mathbf{E}] = \sqrt{\mu}\mathbf{H}, \quad \sqrt{\mu}[\mathbf{k}_0, \mathbf{H}] = -\sqrt{\varepsilon'}\mathbf{E}, \tag{4.38}$$

4.3 Fundamental polarizations. Normal waves

Introduce the dimensionless variables and parameters

$$k_0 x \to x, \quad \sqrt{\mu_0/\varepsilon_0}\mathbf{H} \to \mathbf{H}, \quad \mathbf{E} \to \mathbf{E}, \quad k_0^2 = \varepsilon_0 \mu_0 \omega^2,$$

where ε_0 and μ_0 are permittivity and permeability of vacuum. Propagation of electromagnetic waves along a tube (a waveguide) with cross section Ω (a 2-D domain bounded by smooth curve Γ) parallel to the x_3-axis in the cartesian coordinate system x_1, x_2, x_3, $\mathbf{x} = (x_1, x_2, x_3)$, is described by the homogeneous system of Maxwell equations (written in the normalized form) with the electric and magnetic field dependence $e^{i\gamma x_3}$ on longitudinal coordinate x_3 (the time factor $e^{i\omega t}$ is omitted):

$$\begin{aligned}
&\operatorname{rot}\mathbf{E} = -i\mathbf{H}, \quad \mathbf{x} \in \Sigma,\\
&\operatorname{rot}\mathbf{H} = i\varepsilon\mathbf{E},\\
&\mathbf{E}(\mathbf{x}) = \left(E_1(\mathbf{x}')\mathbf{e}_1 + E_2(\mathbf{x}')\mathbf{e}_2 + E_3(\mathbf{x}')\mathbf{e}_3\right)e^{i\gamma x_3},\\
&\mathbf{H}(\mathbf{x}) = \left(H_1(\mathbf{x}')\mathbf{e}_1 + H_2(\mathbf{x}')\mathbf{e}_2 + H_3(\mathbf{x}')\mathbf{e}_3\right)e^{i\gamma x_3},\\
&\quad \mathbf{x}' = (x_1,\, x_2),
\end{aligned} \tag{4.39}$$

with the boundary conditions for the tangential electric field components on the perfectly conducting surfaces

$$\mathbf{E}_\tau|_M = 0, \tag{4.40}$$

Write system of Maxwell equations (4.39) componentwise

$$\frac{\partial H_3}{\partial x_2} - i\gamma H_2 = i\varepsilon E_1, \quad \frac{\partial E_3}{\partial x_2} - i\gamma E_2 = -iH_1, \quad i\gamma H_1 - \frac{\partial H_3}{\partial x_1} = i\varepsilon E_2,$$

$$i\gamma E_1 - \frac{\partial E_3}{\partial x_1} = -iH_2, \quad \frac{\partial H_2}{\partial x_1} - \frac{\partial H_1}{\partial x_2} = i\varepsilon E_3, \quad \frac{\partial E_2}{\partial x_1} - \frac{\partial E_1}{\partial x_2} = -iH_3,$$

and express functions E_1, H_1, E_2, and H_2 via E_3 and H_3 from the first, second, fourth, and fifth equalities, denoting $\tilde{k}^2 = \varepsilon - \gamma^2$,

$$E_1 = \frac{i}{\tilde{k}^2}\left(\gamma\frac{\partial E_3}{\partial x_1} - \frac{\partial H_3}{\partial x_2}\right), \quad E_2 = \frac{i}{\tilde{k}^2}\left(\gamma\frac{\partial E_3}{\partial x_2} + \frac{\partial H_3}{\partial x_1}\right), \tag{4.41}$$

$$H_1 = \frac{i}{\tilde{k}^2}\left(\varepsilon\frac{\partial E_3}{\partial x_2} + \gamma\frac{\partial H_3}{\partial x_1}\right), \quad H_2 = \frac{i}{\tilde{k}^2}\left(-\varepsilon\frac{\partial E_3}{\partial x_1} + \gamma\frac{\partial H_3}{\partial x_2}\right).$$

Note that this representation is possible if $\gamma^2 \neq \varepsilon_1$ and $\gamma^2 \neq \varepsilon_2$.

It follows from (4.41) that the field of a normal wave can be expressed via two scalar functions

$$\Pi(x_1,x_2) = E_3(x_1,x_2), \quad \Psi(x_1,x_2) = H_3(x_1,x_2).$$

If to look for particular solutions with $E_3 \equiv 0$ then we have a separate problem for the set of component functions $[E_1, E_2, H_3]$, $[H_1, H_2, 0]$ which are called *TE-waves* (transverse electric) or the case of H-polarization. For particular solutions with $H_3 \equiv 0$ we have a problem for the set of component functions $[H_1, H_2, E_3]$, $[E_1, E_2, 0]$ called *TM-waves* (transverse magnetic) or the case of E-polarization. These two cases constitute two fundamental polarizations of the electromagnetic field associated with a given direction of propagation.

For $\gamma = 0$ when we consider fields independent of one of the coordinates (x_3) we have two separate problems for the sets of component functions $[E_1, E_2, H_3]$, TE-(H)polarization, and $[H_1, H_2, E_3]$, TM-(E)polarization.

Thus the problem on normal waves is reduced to boundary eigenvalue problems for functions Π and Ψ. Namely, from (4.39) and (4.40) we have the following eigenvalue problem on normal waves in a waveguide with homogeneous filling: to find $\gamma \in C$, called eigenvalues of normal waves such that there exist nontrivial solutions of the Helmholtz equations

$$\Delta\Pi + \tilde{k}^2\Pi = 0, \quad \mathbf{x}' = (x_1, x_2) \in \Omega \tag{4.42}$$

$$\Delta\Psi + \tilde{k}^2\Psi = 0, \quad \tilde{k}^2 = \varepsilon - \gamma^2, \tag{4.43}$$

satisfying the boundary conditions on Γ_0

$$\Pi|_{\Gamma_0} = 0, \quad \left.\frac{\partial\Psi}{\partial n}\right|_{\Gamma_0} = 0, \tag{4.44}$$

In fact, it is necessary to determine only one function, H_3 for the TE-polarization or E_3 for the TM-polarization; the remaining components are obtained using differentiation.

These problems are considered in more detail in Section 7.1 using vector polarization potentials.

4.4 Problems

4.4.1 Problem

Prove that the Helmholtz equation $\Delta u + \kappa^2 u = 0$ has a solution in the form of plane wave (4.31)

$$E_\alpha = E_\alpha^0 e^{i(\kappa_x x + \kappa_y y + \kappa_z z)}, \quad \kappa_x^2 + \kappa_y^2 + \kappa_z^2 = \kappa^2 \quad (\alpha = x, y, z).$$

4.4.2 Problem

Prove that

$$\operatorname{div}\mathbf{E} = \operatorname{div}(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}}) = ie^{i\mathbf{k}\cdot\mathbf{r}}\mathbf{k}\cdot\mathbf{E}_0 = 0$$

under the condition $\operatorname{div}\mathbf{E}_0 = 0$. For the definitions of vectors see Section 4.2.

4.4.3 Problem

Prove that

$$\operatorname{rot}(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}}) = [\operatorname{grad} e^{i\mathbf{k}\cdot\mathbf{r}}, \mathbf{E}_0],$$

and

$$\sqrt{\varepsilon'}[\mathbf{k}_0, \mathbf{E}_0] = \sqrt{\mu}\mathbf{H}_0,$$

where $\operatorname{rot}\mathbf{E} = i\omega\mu\mathbf{H}$ and $\mathbf{k}_0 = \mathbf{k}/|\mathbf{k}|$ is the unit vector in the direction of the wave propagation.

4.4.4 Problem

Let

$$\mathbf{E} = \operatorname{grad}\operatorname{div}\mathbf{P} + k^2\mathbf{P}, \quad \mathbf{H} = -ik\operatorname{rot}\mathbf{P}, \quad \mathbf{P} = [0, 0, \Pi].$$

Find an explicit expression for all components of vector product $[\mathbf{E}, \mathbf{H}^*]$, where * denotes complex conjugation, in terms of function $\Pi(\mathbf{x})$.

4.4.5 Problem

Prove formulas (4.41).

5. BVPs for Helmholtz equation: classical theory

5.1 Fundamental solutions of the Helmholtz equation, 3D-case

The potential theory developed for the Laplace equation can be extended to the Helmholtz equation

$$\mathscr{L}(c)u := (\Delta + c)u = 0. \tag{5.1}$$

In order to construct fundamental solutions consider, in spherical coordinates, a solution $v_0 = v_0(r)$ depending only on r; the Laplace operator has the form

$$\Delta v_0 = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dv_0}{dr}\right) = \frac{1}{r}\frac{d^2(rv_0)}{dr^2}, \tag{5.2}$$

which yields an ordinary differential equation

$$\frac{d^2w}{dr^2} + cw = 0, \quad w = v_0 r. \tag{5.3}$$

Its linearly independent solutions are

$$\frac{e^{ikr}}{r}, \quad \frac{e^{-ikr}}{r} \quad (c \;=\; k^2 > 0), \tag{5.4}$$

$$\frac{e^{-\kappa r}}{r}, \quad \frac{e^{\kappa r}}{r} \quad (c \;=\; -\kappa^2 < 0). \tag{5.5}$$

The fundamental solution

$$\phi_0(r) = \frac{e^{-ikr}}{r} \tag{5.6}$$

corresponds to an outgoing spherical wave

$$u_0(r) = \frac{e^{i(\omega t - kr)}}{r} \tag{5.7}$$

propagating off a source placed in the origin $r = 0$ where $\phi_0(r)$ has a singularity $\sim \frac{1}{r}$.

Another solution

$$v_0(r) = \frac{e^{ikr}}{r} \tag{5.8}$$

corresponds to an incoming spherical wave

$$u_0(r) = \frac{e^{i(\omega t + kr)}}{r} \tag{5.9}$$

propagating from a source at infinity. This solution is ignored because it has no direct physical sense.

5.2 Behavior of wave fields at infinity (2D-case)

In the two-dimensional case, the Helmholtz equation $\mathscr{L}(k^2)u = 0$ written in the polar coordinates $\mathbf{r} = (r, \phi)$ has the form

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \phi^2} + k^2 u = 0. \tag{5.10}$$

Assume that the function $u = u(\mathbf{r})$ satisfies the Helmholtz equation outside a circle of radius r_0. On any circle of radius $r > r_0$ function u can be decomposed in a trigonometric Fourier series

$$u(\mathbf{r}) = \sum_{n=-\infty}^{\infty} u_n(r) e^{in\phi} \quad (0 < \phi < 2\pi), \tag{5.11}$$

where the coefficients

$$u_n(r) = \frac{1}{2\pi}\int_0^{2\pi} u(\mathbf{r}) e^{-in\phi} d\phi \tag{5.12}$$

are functions of r. In order to find $u_n(r)$ multiply equations (5.10) by $\frac{1}{2\pi}e^{-in\phi}$ and integrate over a circle of radius r. As a result of integration, we obtain

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{du_n}{dr}\right) - \frac{n^2}{r^2}u_n + k^2 u_n = 0, \quad n = 0, \pm 1, \ldots. \tag{5.13}$$

(5.13) is a second-order ordinary differential equation with constant coefficients for $u_n(r)$ which holds for $r > r_0$. Equation (5.13) is actually *the Bessel equation of order n*. Its general solution can be written as

$$u_n(r) = A_n H_n^{(1)}(kr) + B_n H_n^{(2)}(kr), \tag{5.14}$$

where $H_n^{(1,2)}(z)$ are its linearly independent solutions; they are the nth-order Hankel functions of the first and second kind, respectively.

Thus any solution $u = u(\mathbf{r})$ to the homogeneous Helmholtz equation (satisfied outside a circle of radius r_0) can be represented for $r > r_0$ in the form of a series

$$u(\mathbf{r}) = \sum_{n=-\infty}^{\infty} [A_n H_n^{(1)}(kr) + B_n H_n^{(2)}(kr)] e^{in\phi} \quad (0 < \phi < 2\pi,\ r > r_0). \tag{5.15}$$

At infinity, the following asymptotical formulas are valid

$$H_n^{(1,2)}(z) = \sqrt{\frac{2}{\pi z}} e^{\pm i\left(z - \frac{\pi n}{2} - \frac{\pi}{4}\right)} + O\left(\frac{1}{z^{3/2}}\right), \tag{5.16}$$

which yields an asymptotic estimate of the solution to the homogeneous Helmholtz equation at infinity

$$u(\mathbf{r}) = O\left(\frac{1}{\sqrt{r}}\right). \tag{5.17}$$

For the zero-order Hankel functions of the first and second kind, respectively, the following asymptotical formulas are valid

$$\begin{aligned} H_0^{(1)}(z) &= \sqrt{\frac{2}{\pi z}} e^{i\left(z-\frac{\pi}{4}\right)} + \ldots, \\ H_0^{(2)}(z) &= \sqrt{\frac{2}{\pi z}} e^{-i\left(z-\frac{\pi}{4}\right)} + \ldots, \end{aligned} \tag{5.18}$$

5.3 Fundamental solutions of the Helmholtz equation, 2D-case

In the three-dimensional case, fundamental solutions are expressed in terms of elementary functions. According to the analysis in the previous section, the situation is different in the two-dimensional case. In fact, in polar coordinates, the equation for the solution $v_0 = v_0(r)$ of the Helmholtz equation $\mathscr{L}(k^2)u = 0$ depending only on r has the form

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dv_0}{dr}\right) + k^2 v_0 = 0, \tag{5.19}$$

or

$$\frac{d^2 v_0}{dr^2} + \frac{1}{r}\frac{dv_0}{dr} + k^2 v_0 = 0. \tag{5.20}$$

Equation (5.20) is the Bessel equation of zeroth order. Its general solution can be written as

$$v_0(r) = C_1 H_0^{(1)}(kr) + C_2 H_0^{(2)}(kr), \tag{5.21}$$

where the linearly independent solutions

$$\begin{aligned} H_0^{(1)}(z) &= -\frac{2i}{\pi}\ln\frac{1}{z} + g_0^{(1)}(z), \\ H_0^{(2)}(z) &= \frac{2i}{\pi}\ln\frac{1}{z} + g_0^{(2)}(z), \end{aligned} \tag{5.22}$$

where $g_0^{(1,2)}(z)$ are continuously differentiable at the origin, are the zero-order Hankel functions of the first and second kind, respectively. At infinity, the following asymptotical formulas are valid

$$\begin{aligned} H_0^{(1)}(z) &= \sqrt{\frac{2}{\pi z}} e^{i\left(z-\frac{\pi}{4}\right)} + \ldots, \\ H_0^{(2)}(z) &= \sqrt{\frac{2}{\pi z}} e^{-i\left(z-\frac{\pi}{4}\right)} + \ldots, \end{aligned} \tag{5.23}$$

Thus, in the two-dimensional case, the Helmholtz equation has two fundamental solutions

$$\phi_0(r) = H_0^{(1)}(kr) \quad \text{or} \quad \phi_0(r) = H_0^{(2)}(kr) \tag{5.24}$$

corresponding to outgoing cylindrical waves (and to fundamental solutions $\frac{e^{ikr}}{r}$ or $\frac{e^{-ikr}}{r}$ in the three-dimensional case).

The choice of a particular fundamental solution is governed by the chosen time dependence: if the time dependence is $e^{i\omega t}$ or $e^{-i\omega t}$, then $H_0^{(2)}(kr)$ or, respectively, $H_0^{(1)}(kr)$ specifies an outgoing cylindrical wave.

5.4 Integral representation of solution. Potentials

Using notation (5.1) we can write the second Green formula for the Helmholtz operator $\mathscr{L}$ and a domain T bounded by a piecewise smooth surface Σ

$$\int_T [u\mathscr{L}v - v\mathscr{L}u]d\tau = \int_\Sigma \left(u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu} \right) d\sigma. \tag{5.25}$$

Prove this formula following the proof of Section 3.2.

Substituting instead of v a fundamental solution to the Helmholtz equation in the case of three dimensions and repeating literally the proof applied for obtaining an integral representation for a solution to the Poisson equation $\Delta u = -f$ in Section 3.2 (the third Green formula), we arrive at the integral representation of solution to the inhomogeneous Helmholtz equation $\mathscr{L}(k^2)u = -f$

$$\begin{aligned} u(\mathbf{r}) &= \frac{1}{4\pi}\int_\Sigma \left[\frac{e^{-ikR}}{R}\frac{\partial u}{\partial \nu} - u\frac{\partial}{\partial \nu}\left(\frac{e^{-ikR}}{R} \right) \right] d\sigma_{\mathbf{r}_0} + \frac{1}{4\pi}\int_T f(\mathbf{r}_0)\frac{e^{ikR}}{R}d\tau_{\mathbf{r}_0}, \\ R &= |\mathbf{r}-\mathbf{r}_0| = \sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}. \end{aligned} \tag{5.26}$$

One can show that *the volume potentials*

$$v_1(\mathbf{r}) = \frac{1}{4\pi}\int_T f(\mathbf{r}_0)\frac{e^{-ikR}}{R}d\tau_{\mathbf{r}_0}, \quad v_2(\mathbf{r}) = \frac{1}{4\pi}\int_T f(\mathbf{r}_0)\frac{e^{ikR}}{R}d\tau_{\mathbf{r}_0} \tag{5.27}$$

satisfies the inhomogeneous Helmholtz equation $\mathscr{L}(k^2)u = -f$. However, both these functions decay at infinity. This fact dictates the necessity to introduce additional conditions specifying the behavior of solutions to the Helmholtz equation at infinity which would enable one to uniquely determine the solution.

In the 2D-case the volume potentials are

$$v_1(\mathbf{r}) = \int_D f(\mathbf{r}_0)\mathscr{E}(\mathbf{r}-\mathbf{r}_0)dl_{\mathbf{r}_0}, \quad v_2(\mathbf{r}) = \int_D f(\mathbf{r}_0)\frac{i}{4}H_0^{(2)}(kR)dl_{\mathbf{r}_0}, \tag{5.28}$$

where the fundamental solution

$$\mathscr{E}(\mathbf{r}-\mathbf{r}_0) = \frac{i}{4}H_0^{(1)}(k|\mathbf{r}-\mathbf{r}_0|) = \frac{1}{2\pi}\ln\frac{1}{|\mathbf{r}-\mathbf{r}_0|} + h(k|\mathbf{r}-\mathbf{r}_0|)$$

of the Helmholtz operator $\mathscr{L}(k^2)$ is taken in the form (3.17); here $h(z)$ is a continuously differantiable function.

The single layer and double layer potentials (3.15) associated with the Helmholtz equation in the 2D-case are

$$u(\mathbf{r}) = \int_C \mathscr{E}(\mathbf{r}-\mathbf{r}_0)\xi(\mathbf{r}_0)dl_{\mathbf{r}_0}, \quad v(\mathbf{r}) = \int_C \frac{\partial}{\partial \mathbf{n}_{\mathbf{r}_0}}\mathscr{E}(\mathbf{r}-\mathbf{r}_0)\eta(\mathbf{r}_0)dl_{\mathbf{r}_0}, \tag{5.29}$$

where C is a smooth contour. It is easy to verify that potentials (5.29) are in fact generalized single layer and double layer potentials according to Definition 3.4.2. Therefore, for them all the properties and statements proved in Section 3.4.1 are valid.

5.5 Problems for the Helmholtz equation in bounded domains

Formulate the interior Dirichlet problem for the Helmholtz equation: find a function u continuous in $\bar{D} = D \cup \Gamma$ that satisfies the Helmholtz equation in a domain D bounded by the closed smooth contour Γ,

$$\mathscr{L}(k^2)u = \Delta u + k^2 u = 0 \quad \text{in} \quad D, \tag{5.30}$$

and the Dirichlet boundary condition

$$u|_\Gamma = -f, \tag{5.31}$$

where f is a given continuous function.

Formulate the interior Neumann problem: find a function u continuously differentiable in $\bar{D} = D \cup \Gamma$ that satisfies the Helmholtz equation (5.30) in domain D bounded by the closed smooth contour Γ and the Neumann boundary condition

$$\left.\frac{\partial u}{\partial n}\right|_\Gamma = -g, \tag{5.32}$$

where $\frac{\partial}{\partial n}$ denotes the directional derivative in the direction of unit normal vector $\mathbf{n}$ to the boundary Γ directed into the exterior of Γ and g is a given continuous function.

Let us also formulate *Dirichlet and Neumann boundary eigenvalue problems for the Laplace equation*: find a nontrivial solution $u \in C(\bar{D})$ or, respectively, $u \in C^1(\bar{D})$ to the homogeneous Dirichlet or Neumann BVPs

$$-\Delta u = \lambda u \quad \text{in} \quad D, \quad u|_\Gamma = 0, \tag{5.33}$$

or

$$-\Delta u = \lambda u \quad \text{in} \quad D, \quad \left.\frac{\partial u}{\partial n}\right|_\Gamma = 0, \tag{5.34}$$

that correspond to certain (in general complex) values λ called *eigenvalues*.

It is known that eigenvalues of the Dirichlet and Neumann boundary eigenvalue problems for the Laplace equation in a domain D form the sets $\Lambda_{Dir,Neu} = \{\lambda_n^{D,N}\}_{n=1}^\infty$ of isolated real numbers $\lambda_n^{D,N}$ with the accumulation point at infinity; also, $0 \notin \Lambda_{Dir}$ and $0 \in \Lambda_{Neu}$. The complements $\rho_{Dir,Neu} = \mathrm{C} \setminus \Lambda_{Dir,Neu}$, where C denotes the complex λ-plane, are called resolvent (regular) sets of the Dirichlet or Neumann BVPs for the Laplace equation in D.

5.6 Normal waves

Going back to the problems on normal waves we see that the form of solution in (4.39)

$$\begin{aligned} \mathbf{E}(\mathbf{x}) &= (E_1(\mathbf{x}')\mathbf{e}_1 + E_2(\mathbf{x}')\mathbf{e}_2 + E_3(\mathbf{x}')\mathbf{e}_3)\, e^{i\gamma x_3}, \\ \mathbf{H}(\mathbf{x}) &= (H_1(\mathbf{x}')\mathbf{e}_1 + H_2(\mathbf{x}')\mathbf{e}_2 + H_3(\mathbf{x}')\mathbf{e}_3)\, e^{i\gamma x_3}, \\ \mathbf{x}' &= (x_1,\, x_2), \end{aligned} \tag{5.35}$$

with the dependence $e^{i\gamma x_3}$ on longitudinal coordinate x_3 specify a wave propagating in the positive direction of x_3-axis. Problems on normal waves (4.42)–(4.44) have nontrivial solutions if

$$\tilde{k}^2 = \varepsilon - \gamma^2 = \lambda_n^D \quad \text{or} \quad \tilde{k}^2 = \lambda_n^N, \quad n = 1, 2, \ldots, \tag{5.36}$$

so that the eigenvalues of normal waves

$$\gamma = \gamma_n^D = \sqrt{\varepsilon - \lambda_n^D} \quad \text{or} \quad \gamma = \gamma_n^N = \sqrt{\varepsilon - \lambda_n^N}. \tag{5.37}$$

We have $0 \le \lambda_1^{D,N} \le \lambda_2^{D,N} \le \ldots$; therefore, that are at most finitely many values of γ_n^D and γ_n^N that are real, while infinitely many of them are purely imaginary. Consequently, according to (5.35), there are at most finitely many normal waves that propagate without attenuation (in the positive direction of x_3-axis) and infinitely many decay exponentially.

5.7 Reduction to integral equations

According to the definition, the (interior) Dirichlet or Neumann BVPs (5.30), (5.31) or (5.30), (5.32) for the Helmholtz equation in D have at most one solution if λ is not an eigenvalue; that is, if $\lambda \in \rho_{Dir}(D)$ or $\lambda \in \rho_{Neu}(D)$ is a regular value.

The following statements concerning equivalent reduction of the internal BVPs for the Helmholtz equation to integral equations can be proved similarly to the theorems of Section 3.5.

Theorem 12 Let $D \in \mathrm{R}^2$ be a domain bounded by the closed smooth contour Γ. The double layer potential

$$v(\mathbf{r}) = \int\limits_{\Gamma} \frac{\partial}{\partial n_{\mathbf{r}_0}} \mathscr{E}(\mathbf{r}-\mathbf{r}_0)\varphi(\mathbf{r}_0)dl_{\mathbf{r}_0} \tag{5.38}$$

with a continuous density φ is a solution of the interior Dirichlet problem (5.30), (5.31) provided that $\lambda \in \rho_{Dir}(D)$ is a regular value and φ is a solution of the integral equation

$$\varphi(\mathbf{r}) - 2\int\limits_{\Gamma} \frac{\partial \mathscr{E}(\mathbf{r}-\mathbf{r}_0)}{\partial n_{\mathbf{r}_0}} \varphi(\mathbf{r}_0)dl_{\mathbf{r}_0} = -2f(\mathbf{r}), \quad \mathbf{r} \in \Gamma. \tag{5.39}$$

Theorem 13 Let $D \in \mathrm{R}^2$ be a domain bounded by the closed smooth contour Γ. The single layer potential

$$u(\mathbf{r}) = \int\limits_{\Gamma} \mathscr{E}(\mathbf{r}-\mathbf{r}_0)\psi(\mathbf{r}_0)dl_{\mathbf{r}_0} \tag{5.40}$$

with a continuous density ψ is a solution of the interior Nuemann problem (5.30), (5.32) provided that $\lambda \in \rho_{Neu}(D)$ is a regular value and ψ is a solution of the integral equation

$$\psi(\mathbf{r}) + 2\int\limits_{\Gamma} \frac{\partial \mathscr{E}(\mathbf{r}-\mathbf{r}_0)}{\partial n_{\mathbf{r}}} \psi(\mathbf{r}_0)dl_{\mathbf{r}_0} = 2g(\mathbf{r}), \quad \mathbf{r} \in \Gamma. \tag{5.41}$$

5.8 Problems in unbounded domains

5.8.1 Conditions at infinity

Let us recall first that the plane waves propagation along the x-axis have the form

$$\hat{u} = f\left(t - \frac{x}{a}\right), \qquad \hat{\hat{u}} = f\left(t + \frac{x}{a}\right), \tag{5.42}$$

where $\hat{u}$ and $\hat{\hat{u}}$ are, respectively, the forward wave (propagating in the positive direction of the x-axis) and backward wave (propagating in the negative direction of the x-axis). They satisfy the following first-order partial differential equations

$$\frac{\partial \hat{u}}{\partial x} + \frac{1}{a}\frac{\partial \hat{u}}{\partial t} = 0, \tag{5.43}$$

$$\frac{\partial \hat{\hat{u}}}{\partial x} - \frac{1}{a}\frac{\partial \hat{\hat{u}}}{\partial t} = 0. \tag{5.44}$$

In the stationary mode

$$u = v(x)e^{i\omega t} \tag{5.45}$$

For the amplitude function v these relations take the form

$$\frac{\partial \hat{v}}{\partial x} + ik\hat{v} = 0, \tag{5.46}$$

$$\frac{\partial \hat{\hat{v}}}{\partial x} - ik\hat{\hat{v}} = 0, \tag{5.47}$$

for the forward and backward waves, respectively, where $k = \dfrac{\omega}{a}$.

Spherical waves. If a spherical wave is excited by the sources situated in a bounded part of the space (not at infinity), then at large distances from the source, a spherical wave is similar to a plane wave whose amplitude decays as $\dfrac{1}{r}$. This natural physical assumption leads to a conclusion that the outgoing, respectively, incoming, spherical waves must satisfy the relationships

$$\frac{\partial u}{\partial r} + \frac{1}{a}\frac{\partial u}{\partial t} = o\left(\frac{1}{r}\right), \tag{5.48}$$

$$\frac{\partial u}{\partial r} - \frac{1}{a}\frac{\partial u}{\partial t} = o\left(\frac{1}{r}\right). \tag{5.49}$$

For the amplitude functions in the stationary mode we have

$$\frac{\partial v}{\partial r} + ikv = o\left(\frac{1}{r}\right) \quad \text{for outgoing spherical waves,} \tag{5.50}$$

$$\frac{\partial v}{\partial r} - ikv = o\left(\frac{1}{r}\right) \quad \text{for incoming spherical waves.} \tag{5.51}$$

Let us prove now that at large distances from the source, any outgoing spherical wave decays as $\dfrac{1}{r}$.

1. In the case of a point source at the origin, this statement is trivial because the wave itself has the form

$$u(r,t) = \frac{e^{i(\omega t - kr)}}{r} = v_0(r)e^{i\omega t}, \tag{5.52}$$

so that

$$\frac{\partial v_0}{\partial r} + ikv_0 = o\left(\frac{1}{r}\right). \tag{5.53}$$

Check this relationship.

2. Let a spherical wave be excited by a point source situated at a point $\mathbf{r}_0$. The amplitude of the spherical wave is

$$v_0(\mathbf{r}) = \frac{e^{ikR}}{R}, \quad R = |\mathbf{r} - \mathbf{r}_0| = \sqrt{r^2 + r_0^2 - 2rr_0\cos\theta}. \tag{5.54}$$

Calculating the derivative we obtain

$$\frac{\partial R}{\partial r} = \frac{r - r_0\cos\theta}{R} \sim 1 + O\left(\frac{1}{r}\right) \tag{5.55}$$

and

$$\frac{\partial v_0}{\partial R} + ikv_0 = o\left(\frac{1}{R}\right).$$

in view of (5.53). Next,

$$\frac{\partial v_0}{\partial r} = \frac{\partial v_0}{\partial R}\frac{\partial R}{\partial r} = \frac{\partial v_0}{\partial R}\left(1 + O\left(\frac{1}{r}\right)\right) = \frac{\partial v_0}{\partial R} + o\left(\frac{1}{r}\right)$$

because

$$\frac{\partial v_0}{\partial R} \cdot O\left(\frac{1}{r}\right) = o\left(\frac{1}{r}\right).$$

Finally,

$$\frac{\partial v_0}{\partial r} + ikv_0 + o\left(\frac{1}{r}\right) = o\left(\frac{1}{r}\right) \tag{5.56}$$

what is to be proved.

3. Show that the volume potential

$$v(\mathbf{r}) = \int_T f(\mathbf{r}_0)\frac{e^{-ikR}}{R}d\tau_{\mathbf{r}_0}, \quad R = |\mathbf{r} - \mathbf{r}_0|, \tag{5.57}$$

satisfies condition (5.50). Introducing the notation

$$\mathscr{P}v = \frac{\partial v}{\partial r} + ikv, \tag{5.58}$$

we obtain

$$\mathscr{P}v = \int_T f(\mathbf{r}_0)\mathscr{P}\left(\frac{e^{-ikR}}{R}\right)d\tau_{\mathbf{r}_0} = \int_T f(\mathbf{r}_0)o\left(\frac{1}{r}\right)d\tau_{\mathbf{r}_0} = o\left(\frac{1}{r}\right). \tag{5.59}$$

Volume potential (5.27) is the amplitude of an outgoing wave excited by the sources distributed arbitrarily in a bounded domain T. Also, function v defined by (5.57) satisfies the inhomogeneous Helmholtz equation $\mathscr{L}(k^2)u = -f$ and decays as $\frac{1}{r}$ for $r \to \infty$. In addition, it satisfies the condition

$$\frac{\partial v}{\partial r} + ikv = o\left(\frac{1}{r}\right). \tag{5.60}$$

5.8.2 Uniqueness

Theorem 14 There is one and only one solution to the inhomogeneous Helmholtz equation

$$\mathscr{L}(k^2)v = (\Delta + k^2)v = -f(\mathbf{r}), \tag{5.61}$$

where $f(\mathbf{r})$ is a function with local support, which satisfies the conditions at infinity

$$\begin{aligned} v &= O\left(\frac{1}{r}\right), \\ \frac{\partial v}{\partial r} + ikv &= o\left(\frac{1}{r}\right). \end{aligned} \tag{5.62}$$

Proof. Assuming that there are two different solutions v_1 and v_2 and setting

$$w = v_1 - v_2,$$

we see that w satisfies the homogeneous Helmholtz equation $\mathscr{L}(k^2)w = 0$ and the conditions at infinity (5.62). Let Σ_R be a sphere of radius R (later, we will take the limit $R \to \infty$). Applying the third Green formula to $w(\mathbf{r})$ and the fundamental solution $\phi_0(\mathbf{r}_0) = \frac{e^{-ikR}}{4\pi R}$, $R = |\mathbf{r}_0 - \mathbf{r}|$, we arrive at the integral representation of w at a point $\mathbf{r} \in \Sigma_R$

$$w(\mathbf{r}) = \int_{\Sigma_R}\left[\phi_0(\mathbf{r}_0)\frac{\partial w}{\partial r} - w\frac{\partial}{\partial r}(\phi_0(\mathbf{r}_0))\right]d\sigma_{\mathbf{r}_0}. \tag{5.63}$$

The conditions at infinity (5.62) for $w(\mathbf{r})$ and $\phi_0(\mathbf{r})$ yield

$$\begin{aligned}\phi_0 \frac{\partial w}{\partial r} - w\frac{\partial}{\partial \nu}(\phi_0) &= \phi_0 \left[-ikw + o\left(\frac{1}{r}\right)\right] - \\ &- w\left[-ik\phi_0 + o\left(\frac{1}{r}\right)\right] = \phi_0 o\left(\frac{1}{r}\right) - wo\left(\frac{1}{r}\right) = o\left(\frac{1}{r^2}\right).\end{aligned} \tag{5.64}$$

Therefore,

$$w(\mathbf{r}) = \int_{\Sigma_R} o\left(\frac{1}{r^2}\right) d\sigma_{\mathbf{r}_0} \to 0, \quad R \to \infty. \tag{5.65}$$

This implies $w(\mathbf{r}) = 0$ at any $\mathbf{r} \in \Sigma_R$ and thus at any spatial $\mathbf{r}$.

Conditions (5.62) are called *Sommerfeld radiation conditions*.

In the two-dimensional case the Sommerfeld radiation conditions at infinity take the form

$$\begin{aligned} v &= O\left(\frac{1}{\sqrt{r}}\right), \\ \lim_{r\to\infty} \sqrt{r}\left(\frac{\partial v}{\partial r} + ikv\right) &= 0. \end{aligned} \tag{5.66}$$

5.8.3 Statements of problems of the mathematical diffraction theory. Uniqueness

In this section, we summarize the main results concerning the solvability of two-dimensional problems for the Helmholtz equation. We will limit ourselves to a brief survey, assuming that details can be found in original monographs. We will consider the BVPs that arise in electromagnetics and acoustics when the plane wave diffraction and free oscillations (eigenoscillations) of the electromagnetic or acoustic fields in open and closed cylindrical domains are studied.

Prove first an important property of the solution to the Helmholtz equation in an unbounded two-dimensional domain.

Theorem 15 Let $u_0(\mathbf{r})$ be a solution to the Helmholtz equation satisfied outside a circle of radius r_0. If

$$\lim_{r\to\infty} \int_{C_r} |u|^2 dl = 0, \tag{5.67}$$

where C_r is a circle of radius r, then $u \equiv 0$ for $r > r_0$.

Proof. Any solution $u = u(\mathbf{r})$ to the (homogeneous) Helmholtz equation (satisfied outside a circle of radius r_0) can be represented for $r > r_0$ in the form of series (5.15)

$$u(\mathbf{r}) = \sum_{n=-\infty}^{\infty} u_n(r)e^{in\phi}, \quad u_n(r) = A_n H_n^{(1)}(kr) + B_n H_n^{(2)}(kr) \quad (0 < \phi < 2\pi,\ r > r_0). \tag{5.68}$$

Therefore,

$$\lim_{r\to\infty} \int_{C_r} |u|^2 dl = 2\pi \sum_{n=-\infty}^{\infty} r|u_n(r)|^2. \tag{5.69}$$

If

$$\lim_{r\to\infty} \int_{C_r} |u|^2 dl = 0,$$

then (5.69) yields

$$\lim_{r\to\infty} r|u_n(r)|^2 = 0, \quad n = 0, \pm 1, \pm 2, \ldots. \tag{5.70}$$

Next, according to asymptotical formulas (5.16) for Hankel functions $r|u_n(r)|^2$ are bounded quantities at $r \to \infty$, namely,

$$r|u_n(r)|^2 = rO\left(\frac{1}{r}\right) = O(1), \quad n = 0, \pm 1, \pm 2, \ldots, \tag{5.71}$$

which, together with (5.70), implies

$$A_n = B_n = 0, \quad n = 0, \pm 1, \pm 2, \ldots, \tag{5.72}$$

and, consequently, $u \equiv 0$ for $r > r_0$ in line with representation (5.68).

In the three-dimensional case, a similar statement is valid.

Theorem 16 Let $u_0(\mathbf{r})$ be a solution to the Helmholtz equation satisfied outside a sphere S_{r_0} of radius r_0. If

$$\lim_{r\to\infty} \int_{S_r} |u|^2 ds = 0, \tag{5.73}$$

then $u \equiv 0$ for $r > r_0$.

For the vector solutions of Maxwell equations (4.23) and (4.24), electromagnetic field $\mathbf{E}(\mathbf{r})$, $\mathbf{H}(\mathbf{r})$, the similar statements are valid

Theorem 17 Let $\mathbf{E}(\mathbf{r})$, $\mathbf{H}(\mathbf{r})$ be a solution to the Maxwell equation system satisfied outside a sphere of radius r_0. If

$$\lim_{r\to\infty} \int_{S_r} |[\mathbf{H}, \mathbf{e}_r]|^2 ds = 0, \tag{5.74}$$

or

$$\lim_{r\to\infty} \int_{S_r} |[\mathbf{E}, \mathbf{e}_r]|^2 ds = 0, \tag{5.75}$$

where S_r is a sphere of radius r and $\mathbf{e}_r = \mathbf{r}/r$ is the unit position vector of the points on S_r, then $\mathbf{E}(\mathbf{r}) \equiv 0$, $\mathbf{H}(\mathbf{r}) \equiv 0$ for $r > r_0$.

The proof of this theorem is based on expansions of the electromagnetic field in spherical harmonics which are particular solutions to the Maxwell equations in the three-dimensional space and application of asymptotical properties of the spherical harmonics.

5.9 Scalar problem of diffraction by a transparent body

Formulate a scalar (acoustical) problem of the wave diffraction by a transparent body Ω_1. Let Ω_1 be a domain bounded by a piecewise smooth surface Σ. The problem under consideration is reduced to a BVPs for the inhomogeneous Helmholtz equation with a piecewise constant coefficient

$$\begin{aligned} \Delta u_0(\mathbf{r}) + k_0^2 u_0(\mathbf{r}) &= -f_0, \quad \mathbf{r} \in \Omega_0 = R^3 \setminus \overline{\Omega_1}, \\ \Delta u_1(\mathbf{r}) + k_1^2 u_1(\mathbf{r}) &= -f_1, \quad \mathbf{r} \in \Omega_1; \end{aligned} \tag{5.76}$$

solution u satisfies the conjugation conditions on Σ

$$u_1 - u_0 = 0, \qquad \frac{\partial u_1}{\partial n} - \frac{\partial u_0}{\partial n} = 0, \tag{5.77}$$

and the conditions at infinity

$$\begin{aligned} u_0 &= O\left(\frac{1}{r}\right), \\ \frac{\partial u_0}{\partial r} - ik_0 u_0 &= o\left(\frac{1}{r}\right). \end{aligned} \tag{5.78}$$

Theorem 18 The solution to problem (5.76)–(5.78) is unique.

Proof. Since problem (5.76)–(5.78) is linear, it is sufficient to prove that the corresponding homogeneous problem (with $f_0 = f_1 = 0$) has only a trivial solution. Together with u_0 and u_1 consider the corresponding complex conjugate functions u_0^* and u_1^*. They satisfy the same boundary and transmission conditions; however, the condition at infinity takes the form

$$\frac{\partial u_0^*}{\partial r} + ik_0 u_0^* = o\left(\frac{1}{r}\right). \tag{5.79}$$

Applying the second Green formula to u_1^* and u_1^* in domain Ω_1, we obtain

$$\int_\Sigma \left[u_1 \frac{\partial u_1^*}{\partial \nu} - u_1^* \frac{\partial u_1}{\partial \nu}\right] d\sigma_{\mathbf{r}_0} = 0, \tag{5.80}$$

where ν denotes the unit normal vector to the boundary Σ directed into the exterior of Ω_1. Let S_R be a sphere of sufficiently large radius R containing domain Ω_1. Applying the second Green formula to u_0 and u_0^* in the domain Ω_S situated between Ω_1 and S_R, we obtain

$$\int_\Sigma \left[u_0 \frac{\partial u_0^*}{\partial \nu_0} - u_0^* \frac{\partial u_0}{\partial \nu_0}\right] d\sigma_{\mathbf{r}_0} + \int_{S_R} \left[u_0 \frac{\partial u_0^*}{\partial r} - u_0^* \frac{\partial u_0}{\partial r}\right] d\sigma_{\mathbf{r}_0} = 0, \tag{5.81}$$

where $\partial\partial\nu_0$ denotes the directional derivative in the direction of the unit normal vector ν to Σ directed into the interior of Ω_1 (external with respect to Ω_0). Adding up (5.80) and (5.81) and taking into account the conjugation conditions on Σ, we have

$$\int_{S_R} \left[u_0 \frac{\partial u_0^*}{\partial r} - u_0^* \frac{\partial u_0}{\partial r}\right] d\sigma_{\mathbf{r}_0} = 0. \tag{5.82}$$

Applying the condition at infinity and transferring to the limit $R \to \infty$ in (5.82) we obtain

$$\lim_{R\to\infty} \int_{S_R} |u_0|^2 ds = 0, \tag{5.83}$$

Thus $u_0 \equiv 0$ outside sphere S_R according to Theorem 16. Applying the third Green formula (5.63) in Ω_S we obtain that $u_0 \equiv 0$ in Ω_S. Then applying the third Green formula in Ω_1 we obtain that $u_1 \equiv 0$ in Ω_1. Therefore, homogeneous problem (5.76)–(5.78) has only a trivial solution. The theorem is proved.

5.10 Vector problem of diffraction by a transparent body

Formulate a vector (electromagnetic) problem of the wave diffraction by a transparent body Ω_1. Let Ω_1 be a domain bounded by a piecewise smooth surface Σ and $\Omega_0 = R^3 \setminus \overline{\Omega_1}$. The problem under consideration is reduced to a BVP for the inhomogeneous system of Maxwell equations (4.23) and (4.24) with a piecewise constant coefficient

$$\operatorname{rot} \mathbf{H_j} = -i\omega\varepsilon_j \mathbf{E_j} + \mathbf{J_j}, \quad \operatorname{rot} \mathbf{E_j} = i\omega\mu_j \mathbf{H_j}, \quad j = 0, 1, \tag{5.84}$$

with the transmission conditions stating the continuity of the tangential field components across interface Σ

$$[\mathbf{H}_1, \nu] = [\mathbf{H}_0, \nu], \quad [\mathbf{E}_1, \nu] = [\mathbf{E}_0, \nu], \tag{5.85}$$

and the Silver–Müller radiation conditions at infinity

$$\lim_{r\to\infty} r\left([\mathbf{H}_0, \mathbf{e}_r] - ik_0 \mathbf{E}_0\right) = 0, \quad k_0 = \omega\sqrt{\varepsilon_0\mu_0}, \tag{5.86}$$

where ν is the unit normal vector to Σ, $\mathbf{e}_r = \mathbf{r}/r$ is the unit position vector of the points on S_r and the limit holds uniformly with respect to all directions (specified by $\mathbf{e}_r$). Note that in this case (5.84) can be written equivalently (in every domain where the parameters are constant) as a one vector equation with respect to i.e. $\mathbf{E}(\mathbf{r})$ by eliminating $\mathbf{H}(\mathbf{r})$:

$$\operatorname{rot}\operatorname{rot}\mathbf{E_j} - \omega^2\varepsilon_j\mu_j\mathbf{E_j} = \tilde{\mathbf{J}}_\mathbf{j}, \quad j = 0, 1. \tag{5.87}$$

Theorem 19 The solution to problem (5.84)–(5.86) is unique.

Proof. Since problem (5.84)–(5.86) is linear, it is sufficient to prove that the corresponding homogeneous problem (with $\mathbf{J_j} = 0$) has only a trivial solution. Next, one has to apply Theorem 17 and perform the same steps as in the proof of Theorem 18 using Lorentz lemma instead of the Green formulas.

5.11 Problems

5.11.1 Problem

Prove the second Green formula (5.25) for the Helmholtz operator $\mathscr{L}$ and a domain T bounded by a piecewise smooth surface Σ

$$\int_T [u\mathscr{L}v - v\mathscr{L}u]d\tau = \int_\Sigma \left(u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu}\right) d\sigma$$

following the proof of Section 3.2.

5.11.2 Problem

Prove formula (5.26)

$$\begin{aligned} u(\mathbf{r}) &= \frac{1}{4\pi}\int_\Sigma \left[\frac{e^{-ikR}}{R}\frac{\partial u}{\partial \nu} - u\frac{\partial}{\partial \nu}\left(\frac{e^{-ikR}}{R}\right)\right] d\sigma_{\mathbf{r}_0} + \frac{1}{4\pi}\int_T f(\mathbf{r}_0)\frac{e^{ikR}}{R}d\tau_{\mathbf{r}_0}, \\ R &= |\mathbf{r} - \mathbf{r}_0| = \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}. \end{aligned}$$

following the proof of Section 3.2.

5.11.3 Problem

Prove that the single layer and double layer potentials (5.29)

$$u(\mathbf{r}) = \int\limits_C \mathscr{E}(\mathbf{r}-\mathbf{r}_0)\xi(\mathbf{r}_0)dl_{\mathbf{r}_0}, \quad v(\mathbf{r}) = \int\limits_C \frac{\partial}{\partial \mathbf{n}_{\mathbf{r}_0}}\mathscr{E}(\mathbf{r}-\mathbf{r}_0)\eta(\mathbf{r}_0)dl_{\mathbf{r}_0}, \quad \mathscr{E}(\mathbf{r}-\mathbf{r}_0) = \frac{i}{4}H_0^{(1)}(k|\mathbf{r}-\mathbf{r}_0|),$$

satisfy the Helmholtz equation.

5.11.4 Problem

Apply separation of variables and find eigenvalues λ_n^D and eigenfunctions of the Dirichlet boundary eigenvalue problem (5.33) for the Laplace equation in a rectangle Π_{ab} (see problem 3.6.3). Determine normalized eigenfunctions with respect to the norm generated by the inner product $(f,g) = \int\int_{\Pi_{ab}} fg\,dxdy$ in the space $L_2(\Pi_{ab})$ of square-integrable functions.

5.11.5 Problem

Apply separation of variables and find eigenvalues λ_n^N and eigenfunctions of the Neumann boundary eigenvalue problem (5.34) for the Laplace equation in a rectangle Π_{ab} (see problem 3.6.3).

5.11.6 Problem

Show that eigenfunctions of the Dirichlet boundary eigenvalue problem (5.33) for the Laplace equation in a rectangle Π_{ab} are orthogonal with respect to the inner product introduced in 5.11.1.

5.11.7 Miniproject 1: solution to the Dirichlet problem for the Poisson equation

Solve the BVP in a rectangle $\Pi_{ab} = \{(x,y) : 0 < x < a, 0 < y < b\}$

$$\begin{cases} -\Delta u = F(x,y), \quad u = u(x,y), \quad 0 < x < a,\ 0 < y < b, \qquad u \in C^2(\Pi_{ab}) \cap C(\bar{\Pi}_{ab}) \\ u(0,y) = 0, \qquad u(a,y) = 0, \quad 0 \le y \le b, \\ u(x,0) = 0, \qquad u(x,b) = 0, \quad 0 \le x \le a, \end{cases}$$

$$F(x,y) = \begin{cases} A \sin \frac{\pi}{h_1} \left[x - \left(x_0 - \frac{h_1}{2} \right) \right] \sin \frac{\pi}{h_2} \left[y - \left(y_0 - \frac{h_2}{2} \right) \right], \ (x,y) \in \Pi_{h_1 h_2}(x_0,y_0), \\ 0, \quad (x,y) \notin \Pi_{h_1 h_2}(x_0,y_0), \end{cases} \tag{5.88}$$

with $\quad supp\, F(x,y) = \Pi_{h_1 h_2}(x_0,y_0) \subset \Pi_{ab}$,

$$\Pi_{h_1 h_2}(x_0,y_0) = \left\{ (x,y) : x_0 - \frac{h_1}{2} < x < x_0 + \frac{h_1}{2},\ y_0 - \frac{h_2}{2} < y < y_0 + \frac{h_2}{2} \right\}.$$

Hints: Decompose $F(x,y)$ in Fourier series $\Sigma_{\nu=1}^{\infty} f_\nu \phi_\nu$ in (normalized and orthogonal) eigenfunctions ϕ_ν of the Dirichlet boundary eigenvalue problem (5.33) for the Laplace equation in rectangle Π_{ab}; that is, calculate Fourier coefficients f_ν of $F(x,y)$. Look for the solution u to the BVP in the form of Fourier series $\Sigma_{\nu=1}^{\infty} u_\nu \phi_\nu$ with unknown Fourier coefficients u_ν, find a relation between u_ν and f_ν.

5.11.8 Miniproject 2: example of inverse problem

Prove that in 5.11.7 it is possible, under certain conditions, to uniquely determine any of the five parameters A, x_0, y_0, h_1, h_2 provided that the remaining four are given from the knowledge of one Fourier coefficient $u_1 = u_1(A, x_0, y_0, h_1, h_2)$ of $u(x,y)$.

6. BVPs for Helmholtz equation: pseudodifferential approach

6.1 Formulation of the problems

Denote by Γ the boundaries of the cross sections [in the plane $\mathbf{x} = (x, y)$] of cylindrical domains with the generatrix directed along the z axis and consider the following three typical cases.

(i) Γ is an unclosed nonintersecting finite piecewise smooth curve (a strip) situated in the free space (in particular, strip Γ may coincide with a rectilinear segment or a circular arc. We will also consider a more general case when Γ consists of a finite number of simple, planar, closed or unclosed smooth curves Γ_j of a finite length that belong to class C^∞. Introduce the following notation:

$$\Gamma = \bigcup_j \Gamma_j, \quad \overline{\Gamma}_i \bigcap \overline{\Gamma}_j = \emptyset \ \text{ for } \ i \neq j,$$

where the edges

$$\partial\Gamma = \bigcup_j (\overline{\Gamma}_j \setminus \Gamma_j)$$

are the endpoints of Γ (these points do not belong to Γ and $\partial\Gamma \bigcap \Gamma = \emptyset$). Let G_0 be a union of internal domains bounded by closed curves Γ_j, and we set $G_0 = \emptyset$ if these domains are absent.

(ii) $\Gamma = (\partial\Omega^1 \cup \partial\Omega^2) \setminus \overline{\Sigma}$, where $\partial\Omega^1$ and $\partial\Omega^2$ are the boundaries of the half-plane $\Omega^1 = \Omega^+ = \{\mathbf{x} : y > 0\}$ [which is, of course, an unbounded $S_\Pi(\Sigma)$-domain introduced in Chapter 2] and of a bounded $S_\Pi(\Sigma)$-domain $\Omega^2 \subset \Omega^- = \{\mathbf{x} : y < 0\}$, respectively, having a common part $\Gamma_{12} = (\partial\Omega^2 \cap \partial\Omega^1) \in \{\mathbf{x} : y = 0\}$ containing an interval (a slot) Σ, $\Sigma \subseteq \Gamma_{12}$; function $\varepsilon = \varepsilon(\mathbf{x}) = \varepsilon_n =$const $\mathbf{x} \in \Omega^n, n = 1, 2$ specifies the permittivity of the medium. We indicate a particular case when the $S_\Pi(\Sigma)$-domain $\Omega^2 = \Pi_{ab} = \{\mathbf{x} : -a/2 < x < a/2 \ -b < y < 0\}$ is a rectangle, and $\Sigma = \{\mathbf{x} : y = 0, \ -l < x < l\}$ is the slot with the edges $\partial\Sigma = \{A_1 = -l, \ A_2 = l\}$.

(iii) $\Gamma = (\partial\Pi_{ab_1} \cup \partial\Pi_{ab_2}) \setminus \overline{L}$, where $\Omega^1 = \Pi_{ab_1} = \{\mathbf{x} =: \ 0 < x < a, \ 0 < y < b_1\}$ and $\Omega^2 = \Pi_{ab_2} = \{\mathbf{x} : 0 < x < a, b_2 < y < 0\}$ are two rectangular domains, and $\Sigma = \{\mathbf{x} : y = 0, a/2 - w = d_1 < x < d_2 = a/2 + w\}$ is the slot with the edges $\partial\Sigma = \{A_1 = d_1, \ A_2 = d_2\}$.

In case (i), the acoustic and electromagnetic two-dimensional problems of diffraction by such curved boundaries are reduced to the determination of a scalar function u (the scattered field) that satisfies the homogeneous Helmholtz equation

$$\Delta u + k^2 u = 0 \quad \text{in} \quad \mathrm{R}^2 \setminus (\overline{\Gamma} \bigcup G_0), \tag{6.1}$$

the Dirichlet,

$$u|_\Gamma = -f, \tag{6.2}$$

or the Neumann,

$$\left.\frac{\partial u}{\partial n}\right|_\Gamma = -g, \tag{6.3}$$

boundary conditions on Γ, and the Sommerfeld radiation conditions at infinity

$$u = O(r^{-1/2}), \quad \frac{\partial u}{\partial r} - iku = o(r^{-1/2}) \tag{6.4}$$

for $r = |\mathbf{x}| = (x^2 + y^2)^{1/2} \to \infty$. The Dirichlet problem corresponds to a soft screen in acoustics (or to the case of E-polarization in electrodynamics), and the Neumann problem, to a hard screen (or to the case of H-polarization). Here, k means the free-space wavenumber and $\Im k \geq 0$, $k \neq 0$.

In addition to the above conditions, field u must satisfy the requirement that provides the finiteness of energy in every bounded spatial domain:

$$u \in H^1_{loc}(\mathrm{R}^2 \setminus \overline{G_0}); \tag{6.5}$$

that is,

$$\int\limits_{G \setminus G_0} \left(|\nabla u|^2 + |u|^2 \right) d\mathbf{x} < \infty$$

for every bounded domain G.

We also formulate the BVPs for the homogeneous Helmholtz equation with a piecewise constant coefficient

$$\Delta u(\mathbf{x}) + \lambda \varepsilon(\mathbf{x}) u(\mathbf{x}) = 0, \quad \lambda = k^2, \quad \mathbf{x} \in \Omega, \tag{6.6}$$

where $\Omega = R^2 \setminus S$ in case (i) and $\Omega = \Omega^1 \cup \Omega^2$ in cases (ii) and (iii). For the classical solution, one may require additionally that

$$u \in M = \{u : \; u \in C^2(\Omega) \cap C^1(\cup_i \overline{\Omega}^{\varepsilon_i} \setminus \Gamma_\delta)\},$$

where

$$\Gamma_\delta = \{\mathbf{x} : dist(\mathbf{x}, \partial\Gamma) < \delta\}$$

is a δ-vicinity of the endpoints of curve Γ (edges), and $\Omega^{\varepsilon_j} = \{\mathbf{x} : \varepsilon(\mathbf{x}) = \varepsilon_j = \text{const}\}$; solution u satisfies homogeneous boundary conditions (6.2) or (6.3) on Γ, the conjugation conditions on Σ [in cases (ii) and (iii)]

$$u^+ - u^- = 0, \quad \left[\frac{1}{\varepsilon(\mathbf{x})} \frac{\partial u^+}{\partial n} - \frac{1}{\varepsilon_0} \frac{\partial u^-}{\partial n} \right] = 0, \tag{6.7}$$

or

$$u^+ - u^- = 0, \quad \left[\frac{\partial u^+}{\partial n} - \frac{\partial u^-}{\partial n} \right] = 0; \tag{6.8}$$

in case (ii), when the diffraction of the plane H- or E-polarized wave is considered, the first (homogeneous) conjugation condition may be replaced by the inhomogeneous condition

$$u^+ - u^- = \psi_1, \quad \left[\frac{1}{\varepsilon(\mathbf{x})} \frac{\partial u^+}{\partial n} - \frac{1}{\varepsilon_0} \frac{\partial u^-}{\partial n} \right] = 0, \tag{6.9}$$

or

$$u^+ - u^- = 0, \quad \left[\frac{\partial u^+}{\partial n} - \frac{\partial u^-}{\partial n}\right] = \psi_2, \tag{6.10}$$

where $\psi_j = \psi_j(x)$ $(j = 1, 2)$ is the given differentiable function on Σ and the Meixner (edge) conditions (6.5). For complex λ, the Sommerfeld radiation condition can be replaced by a more general radiation condition: there exists an $R_0 > 0$ such that for all $\mathbf{x} = (r, \phi) : r \geq R_0$, the following representations hold

$$u(\mathbf{x}) = \sum_{m=-\infty}^{+\infty} a_m H_m^{(1)}(kr) p_m(\phi), \tag{6.11}$$

where a_m are the complex numbers, $p_m(\phi) = e^{im\phi}$ $(m = 0, \pm 1, \pm 2, \dots)$ in case (i); in case (ii), $p_m(\phi) = 0$ $(m = -1, -2, \dots)$ and $p_m(\phi) = \sin m\phi$ for (6.2) and $p_m(\phi) = \cos m\phi$ for (6.3); the series in (6.11) admit double termwise differentiation with respect to $r \geq R_0$ and $\phi \in (0, 2\pi)$ [case (i)] or $\phi \in (0, \pi)$ [case (ii)]; the Hankel function $H_m^{(1)}(z)$ of the first kind and mth order may be generally considered on the appropriate Riemann surface of complex variable z. of the analytical continuation with respect to the spectral parameters k or λ of the fundamental solution $\mathscr{E}$ of the two-dimensional Helmholtz operator $\mathscr{H}$; (in electromagnetic diffraction problems, $k = \frac{\omega}{c}$ is referred to as the wavenumber of free space, ω is the frequency parameter, and c is the speed of light in free space). Note that $\mathscr{E}(\mathbf{x} - \mathbf{y})$ satisfies the Sommerfeld radiaton conditions in domain Ω^+ for real values of k.

Homogeneous problems (6.1), (6.2) with $f = 0$, (6.5), and (6.11); (6.1), (6.3) with $g = 0$, (6.5), and (6.11); (6.6), (6.2) with $f = 0$, (6.8), (6.5), and (6.11); and (6.6), (6.3) with $g = 0$, (6.7), (6.5), and (6.11) concerning the determination of (complex) values of spectral parameters k or λ (eigenvalues), for which there exist their nontrivial solutions (in appropriate spaces), will be called *problems E* (with the Dirichlet conditions on Γ) and *problems H* (with the Neumann conditions on Γ).

These problems may be conditionally referred to as the problem on eigenfrequencies of open [(ii)] and shielded [(iii)] slot resonators.

The respective inhomogeneous BVP (6.1), (6.2), (6.4), and (6.5); (6.1), (6.3)–(6.5); (6.6), (6.2), (6.8), (6.4), and (6.5) [or (6.6), (6.2) with $f = 0$, (6.10), (6.4), and (6.5)]; and (6.6), (6.3), (6.7), (6.4), and (6.5) [or (6.6), (6.3) with $g = 0$, (6.9), (6.4), and (6.5)] (that correspond in electromagnetics to the diffraction of the plane H-polarized wave), will be called, respectively, *problems Ei* and *problems Hi*.

The form of the radiation condition (6.11) in these problems is connected with the fact that the spectral parameter may be complex. For real k (when considering the diffraction problem Hi in the lossless medium with $\Im\varepsilon = 0$), one can replace (6.11) by the equivalent Sommerfeld radiation conditions (6.4) or the Kupradze radiation conditions for the complex k with $\Im k > 0$. (6.11) may be considered as a generalization of the latter conditions and can be applied for arbitrary nonzero complex k (or λ). This condition may be also considered as the continuation of the Sommerfeld condition from the real axis of the complex parameter or of the Kupradze condition from the zero-sheet upper half-plane to the appropriate Riemannian surface.

In (6.2) and (6.3), the equality means the equality of elements from spaces $H^{1/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$, and (6.3) must hold on both sides of Γ. It is well known that solutions to the homogeneous Helmholtz equation belonging to $H^1_{loc}(\mathrm{R}^2 \setminus (\overline{\Gamma} \bigcup G_0))$ are infinitely differentiable in $\mathrm{R}^2 \setminus (\overline{\Gamma} \bigcup G_0)$; therefore, one can assume that $u \in C^2(\mathrm{R}^2 \setminus (\overline{\Gamma} \bigcup G_0))$ and understand (6.1) in a usual sense. If the field sources are situated outside the curved boundaries, then functions f and g, which correspond to the trace of the incident field and its normal derivative on Γ, are infinitely differentiable on Γ and $f, g \in C^\infty(\overline{\Gamma})$. Moreover, if $\Gamma' \subset \Gamma$ is a smooth part of a C^∞ curve, then one can show that solution

u is infinitely differentiable up to Γ' (on each side of the curve if $L' \bigcap \partial G_0 = \emptyset$), and

$$u \in C^2(\mathrm{R}^2 \setminus (\overline{\Gamma} \bigcup G_0)) \bigcap_{\delta>0} C^1(\overline{G_1 \setminus (\Gamma_\delta \bigcup G_0)}) \bigcap_{\delta>0} C^1(\overline{G_2 \setminus \Gamma_\delta}),$$

if all Γ_j belong to C^∞.

6.2 Uniqueness and existence theorems

Consider the uniqueness of the problems (6.1, (6.2), (6.4), and (6.5) and (6.1, (6.3)–(6.5). We will prove that for $\Im k \geq 0$ and $k \neq 0$, these Dirichlet and Neumann problems with homogeneous boundary conditions have only trivial solutions. Denote by $[\,\cdot\,]$ the difference of the limiting values of a function taken from different sides of a curve. Equations (6.1)–(6.3) yield the transmission problem for u:

$$\Delta u + k^2 u = 0 \quad \text{in} \quad G_2 \bigcup (G_1 \setminus G_0),$$

$$[u]_{\Lambda\setminus\overline{\Gamma}} = \left[\frac{\partial u}{\partial n}\right]_{\Lambda\setminus\overline{\Gamma}} = 0,$$

$$u|_\Gamma = 0 \quad \text{or} \quad \left.\frac{\partial u}{\partial n}\right|_\Gamma = 0,$$

with the radiation condition (6.4) for $r \to \infty$. Recall that Λ is a closed, smooth C^∞ curve that contains Γ ($\Gamma \subset \Lambda$) and divides the plane into bounded, G_1, and unbounded, G_2, domains, and n is the normal vector external with respect to G_1.

Denote by $B_R = \{\mathbf{x} : |\mathbf{x}| < R\}$ a circle of radius R such that $\Lambda \subset B_R$. Applying the second Green's formula in domains $G_1 \setminus G_0$, $G_2 \bigcap B_R$ to functions u and $\overline{u}$ and adding up the results, we obtain

$$\int_{\partial B_R} \left(\frac{\partial u}{\partial r}\overline{u} - \frac{\partial \overline{u}}{\partial r} u\right) dl = -4ik'k'' \int_{B_R} |u|^2\, d\mathbf{x}, \quad k = k' + ik''.$$

Integrals over Λ vanish by virtue of the boundary and transmission conditions. Here, the applicability of the Green's formula follows from the fact that u belongs to $H^1_{loc}(\mathrm{R}^2 \setminus G_0)$ and contour Λ is smooth. A standard analysis of the obtained identity shows that $u \equiv 0$ in G_2. The transmission conditions and the fact that u is an analytical function in $\mathrm{R}^2 \setminus (\overline{\Gamma} \bigcup G_0)$ yield $u \equiv 0$ in $G_1 \setminus G_0$. Thus, the uniqueness of the solution to problem (6.1)–(6.5) is proved.

We will look for the unique solution to problem (6.1)–(6.5) in the form of potentials

$$u(\mathbf{x}) = K_0(q\varphi) = -\frac{i}{4}\int_\Gamma H_0^{(1)}(k|\mathbf{x}-\mathbf{y}|)\varphi(\mathbf{y})\,dl, \quad \varphi \in \tilde{H}^{-1/2}(\Gamma) \tag{6.12}$$

for the Dirichlet problem and

$$u(\mathbf{x}) = K_1(q\psi) = \frac{i}{4}\int_\Gamma \frac{\partial}{\partial n_y} H_0^{(1)}(k|\mathbf{x}-\mathbf{y}|)\psi(\mathbf{y})\,dl, \quad \psi \in \tilde{H}^{1/2}(\Gamma) \tag{6.13}$$

for the Neumann problem. Here, $H_0^{(1)}(z)$ is the Hankel function of the first kind. Continuing φ and ψ as zero functions from Γ to Λ, performing the restriction on Γ, and taking into account the continuity of operators q and p, we prove that

$$\varphi = \left[\frac{\partial u}{\partial n}\right]_\Gamma \text{ and } [u]_\Gamma = 0 \text{ for the Dirichlet problem}$$

and

$$\psi = [u]_\Gamma \text{ and } \left[\frac{\partial u}{\partial n}\right]_\Gamma = 0 \text{ for the Neumann problem.}$$

In addition to this,

$$u \in H^1_{loc}(\mathrm{R}^2 \setminus \overline{G_0})$$

for all

$$\varphi \in \tilde{H}^{-1/2}(\Gamma), \ \psi \in \tilde{H}^{1/2}(\Gamma)$$

[see (6.12) and (6.13)]. Here,

$$[u]_\Gamma = p\left(\gamma_0\left(u|_{G_2}\right) - \gamma_0\left(u|_{G_1}\right)\right),$$

$$\left[\frac{\partial u}{\partial n}\right]_\Gamma = p\left(\gamma_1\left(u|_{G_2}\right) - \gamma_1\left(u|_{G_1}\right)\right).$$

Since solution u is chosen in the form of potentials (6.12) and (6.13), equation (6.1) and conditions (6.4) and (6.5) hold for all φ and ψ. Thus, we have to satisfy only boundary conditions (6.2) or (6.3), which yields the equations on curves Γ:

$$D\varphi = \frac{i}{4}\int\limits_\Gamma H_0^{(1)}(k|\mathbf{x}-\mathbf{y}|)\varphi(\mathbf{y})\,dl = f(\mathbf{x}), \quad \mathbf{x} \in \Gamma, \tag{6.14}$$

$$D = p\gamma_0 K_0 q,$$

for the Dirichlet problem, and

$$N\psi = \frac{i}{4}\frac{\partial}{\partial n_\mathbf{x}}\int\limits_\Gamma \frac{\partial}{\partial n_\mathbf{y}} H_0^{(1)}(k|\mathbf{x}-\mathbf{y}|)\psi(\mathbf{y})\,dl = -g(\mathbf{x}), \quad \mathbf{x} \in \Gamma, \tag{6.15}$$

$$N = p\gamma_1 K_1 q,$$

for the Neumann problem. The first equation has a weak (logarithmic) singularity and the second equation is hypersingular. Using the methods of the analysis of logarithmic integral operators in the Sobolev spaces in terms of PDOs (see Section 1.8), we can consider both equations (6.14) and (6.15) from a single viewpoint as pseudodifferential equations.

Thus, if φ and ψ are solutions to (6.14) and (6.15), formulas (6.12) and (6.13) give the solution to diffraction problem (6.1)–(6.5).

Now, we consider the case when all curves Γ_j are closed and smooth. Then,

$$D : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$$

and

$$N : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$$

are the Fredholm operators with zero indices. In addition, D and N are elliptic PDOs of orders -1 and $+1$, respectively.

By virtue of the regularity theorem, if the equations

$$D\varphi = f \tag{6.16}$$

and

$$N\psi = -g \tag{6.17}$$

have C^∞ right-hand sides and Γ is a C^∞ curve, then solutions φ and ψ on Γ belong to C^∞. Equations (6.16) and (6.17) or (6.14) and (6.15) are uniquely solvable for all k: $\Im k \geq 0$, $k \neq 0$ except for a discrete set of points. According to the definitions of Section 1.6, these points are the CNs and they are situated on the real axis with the only possible accumulation point at infinity. For equations (6.16) and (6.17), all CNs are isolated and have finite algebraic multiplicity. We will denote by $\sigma(D)$ and $\sigma(N)$ the sets of CNs for equations (6.16) and (6.17), respectively.

When the Dirichlet or the Neumann problem is solved for $k \in \sigma(D)$ or $k \in \sigma(N)$, (6.16) or (6.17) should be replaced by modified equations that are uniquely solvable for $k \in \sigma(D)$ or $k \in \sigma(N)$. Thus, for all values k such that $\Im k \geq 0$ and $k \neq 0$, the Dirichlet and the Neumann problems (6.1)–(6.5) are uniquely solvable. However, the solutions can be represented in the form of potentials (6.12) and (6.13) only if $k \notin \sigma(D)$ for the Dirichlet problem and $k \notin \sigma(N)$ for the Neumann problem. We have proved the following statement.

Theorem 20 If $k \notin \sigma(D)$, then the operator

$$D : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$$

is bounded and has a bounded inverse. In this case, equation (6.16) is uniquely solvable. If $k \notin \sigma(N)$, then the operator

$$N : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$$

is bounded and has a bounded inverse. In this case, equation (6.17) is uniquely solvable.

Assume that all Γ_j are unclosed smooth curves. Before to proceed to the analysis of the general case of the diffraction problem on Γ, we consider a particular case of intervals situated on one line, when

$$\Gamma = \left\{ \mathbf{x} : y = 0, a_{2j-1} < x < a_{2j}, j = 1, \dots, J; a_i < a_j (i < j) \right\}.$$

In this case, D and N are convolution-type operators on the line R^1. Calculating the Fourier transforms of the kernels of these operators and using the convolution theorem and the properties of the Fourier transform, we rewrite equations (6.14) and (6.15) as

$$D\varphi \equiv \int_{-\infty}^{+\infty} \frac{1}{\sqrt{\xi^2 - k^2}} e^{ix\xi} \widehat{\varphi}(\xi)\, d\xi = f(x), \quad x \in \Gamma, \tag{6.18}$$

and

$$N\psi \equiv \int_{-\infty}^{+\infty} \sqrt{\xi^2 - k^2} e^{ix\xi} \widehat{\psi}(\xi)\, d\xi = g(x), \quad x \in \Gamma. \tag{6.19}$$

Here, $\widehat{\varphi}$ and $\widehat{\psi}$ denote the Fourier transforms of φ and ψ. We will look for solutions φ and ψ in the spaces of generalized functions (distributions) $\varphi \in \tilde{H}^{-1/2}(\Gamma)$ and $\psi \in \tilde{H}^{1/2}(\Gamma)$. In the case under consideration, these spaces can be described in terms of the Fourier transform:

$$\tilde{H}^s(\Gamma) = \left\{ u : (1 + |\xi|)^s \widehat{u}(\xi) \in L_2(\mathrm{R}^1), \operatorname{supp} u \subset \overline{\Gamma} \right\}.$$

The left-hand sides of equations (6.18) and (6.19) define the boundary integrodifferential operators

$$D : \tilde{H}^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$$

and

$$N : \tilde{H}^{1/2}(\Gamma) \to H^{-1/2}(\Gamma),$$

where $H^s(\Gamma)$ is a restriction of $H^s(\mathrm{R}^1)$ on Γ. We have

$$\begin{aligned} \left(\xi^2 - k^2\right)^{-1/2} &= |\xi|^{-1} + O(|\xi|^{-3}), \\ \left(\xi^2 - k^2\right)^{1/2} &= |\xi| + O(|\xi|^{-1}) \end{aligned}$$

for $|\xi| \to \infty$; therefore, D and N are Fredholm PDOs with the zero index and orders -1 and $+1$, respectively. These operators are invertible by virtue of the estimates

$$\left|(D\varphi,\varphi)_{L_2}\right| \geq \frac{1}{\sqrt{2}} \int_{-\infty}^{+\infty} \left|\xi^2 - k^2\right|^{-1/2} |\widehat{\varphi}(\xi)|^2 \, d\xi > 0, \quad \varphi \neq 0,$$

and

$$\left|(N\psi,\psi)_{L_2}\right| \geq \frac{1}{\sqrt{2}} \int_{-\infty}^{+\infty} \left|\xi^2 - k^2\right|^{1/2} |\widehat{\psi}(\xi)|^2 \, d\xi > 0, \quad \psi \neq 0.$$

These estimates prove that the kernels of these operators contain only the zero element: $\ker D = \{0\}$ and $\ker N = \{0\}$. Thus, by the Fredholm alternative, equations (6.18) and (6.19) or (6.14) and (6.15) are uniquely solvable.

Consider the general case of a system of curves of arbitrary shape Γ. The operators

$$D : \tilde{H}^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$$

and

$$N : \tilde{H}^{1/2}(\Gamma) \to H^{-1/2}(\Gamma),$$

are bounded, and their Fredholm property follows from the Gårding inequality

$$\begin{aligned}
\Re\langle (D + K_D)\,\varphi,\varphi\rangle &\geq \lambda_D \,\|\varphi\|^2_{-1/2}, \quad \forall \varphi \in \tilde{H}^{-1/2}(\Gamma),\\
\Re\langle (N + K_N)\,\psi,\psi\rangle &\geq \lambda_N \,\|\varphi\|^2_{1/2}, \quad \forall \psi \in \tilde{H}^{1/2}(\Gamma)
\end{aligned}$$

with certain compact operators

$$\begin{aligned}
K_D : &\quad \tilde{H}^{-1/2}(\Gamma) \to H^{1/2}(\Gamma),\\
K_N : &\quad \tilde{H}^{1/2}(\Gamma) \to H^{-1/2}(\Gamma).
\end{aligned}$$

Equations (6.14) and (6.15) are uniquely solvable for arbitrary right-hand sides $f \in H^{1/2}(\Gamma)$ or $g \in H^{-1/2}(\Gamma)$, since the corresponding homogeneous equations have only trivial solutions. Indeed, if we assume that φ and ψ are nontrivial solutions of (6.14) and (6.15) for $f = 0$ and $g = 0$, then formulas (6.12) and (6.13) yield nontrivial solutions of homogeneous problems (6.1)–(6.5), which contradicts the uniqueness theorem. Thus, in the case of unclosed curves, equations (6.16) and (6.17) has a unique solution for all k: $\Im k \geq 0$, $k \neq 0$.

Theorem 21 If $\Im k \geq 0$ and $k \neq 0$, then each operator

$$D : \tilde{H}^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$$

and

$$N : \tilde{H}^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$$

is bounded and has a bounded inverse. Equations (6.16) and (6.17) are uniquely solvable.

When the diffraction problems are considered, it is important to investigate the asymptotic behaviour of solutions in the vicinities of the endpoints of $\partial\Gamma$. Let f and g be smooth functions, e.g., $f, g \in C^\infty(\overline{\Gamma})$. Applying the known results concerning the regularity of solutions of elliptic equations, one can easily show that φ and ψ are smooth functions on Γ (they belong to C^∞ if all $\Gamma_j \in C^\infty$) and have singularities

$$\varphi = O(\rho^{-1/2}), \quad \rho \to 0, \tag{6.20}$$

$$\psi = O(\rho^{1/2}), \quad \rho \to 0, \tag{6.21}$$

in the vicinities of the endpoints of $\partial\Gamma$ (ρ is the distance to an endpoint of the curve); note that estimates (6.20) and (6.21) are accurate with respect to the order.

7. Wave propagation and diffraction in guides

7.1 Wave propagation in a guide

We have already introduced normal waves as solutions (5.35) to homogeneous Maxwell equations with the imposed dependence $e^{i\gamma x_3}$ on longitudinal coordinate x_3. Let us show that the solutions in the form of normal waves can be obtained independently from a different viewpoint

To this end, consider again a tube (a waveguide) parallel to the x_3-axis in the cartesian coordinate system $x_1,\ x_2,\ x_3$, $\mathbf{x} = (x_1,\ x_2,\ x_3)$ bounded by a smooth cylindrical surface Σ with transversal (by the plane $x_3 =$const) cross section Ω (a 2-D domain bounded by smooth curve Γ) filled with a homogeneous medium having permittivity and permeability ε and μ; denote by $k = \omega\sqrt{\mu\varepsilon}$ the wavenumber of the medium. Propagation of electromagnetic waves along the waveguide is described by the homogeneous system of Maxwell equations which can be written in the form

$$\begin{aligned} \operatorname{rot}\mathbf{H} &= -ik\mathbf{E}, \\ \operatorname{rot}\mathbf{E} &= ik\mathbf{H}, \end{aligned} \tag{7.1}$$

with the boundary conditions for the tangential electric field components on the perfectly conducting walls Σ of the waveguide

$$\mathbf{E}_\tau|_\Sigma = 0, \tag{7.2}$$

Look for particular solutions of (7.1) in the form

$$\begin{aligned} \mathbf{E} &= \operatorname{grad}\operatorname{div}\mathbf{P} + k^2\mathbf{P}, \\ \mathbf{H} &= -ik\operatorname{rot}\mathbf{P}, \end{aligned} \tag{7.3}$$

using the polarization potential $\mathbf{P} = [0,0,\Pi]$ that has only one nonzero component $P_3 = \Pi$. It is easy to see that

$$H_3 = 0, \quad \mathbf{E} = [0,0,E_3], \quad \mathbf{H} = [H_1,H_2,0], \tag{7.4}$$

and this case is called TM-polarization or E-polarization Substituting (7.3) into (7.1) yields the equations

$$\begin{aligned} \Delta_3\Pi + k^2\Pi &= 0 \quad \text{or} \quad \Delta\Pi + \frac{\partial^2\Pi}{\partial x_3^2} + k^2\Pi = 0, \\ \Delta_3 &= \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}. \end{aligned} \tag{7.5}$$

Condition (7.2) is satisfied if we assume that

$$\Pi|_\Sigma = 0, \tag{7.6}$$

because the third components of both **P** and **E** are actually tangential components that must vanish on the waveguide wall and they are coupled by the first relation (7.3). (7.5) and (7.6) constitute the Dirichlet BVP for the Helmholtz equation in the tube. We look for the solution to this problem in the form

$$\Pi(\mathbf{x}) = \Pi(\mathbf{x}', x_3) = \psi(\mathbf{x}')f(x_3), \quad \mathbf{x}' = (x_1, x_2), \quad \psi(\mathbf{x}'), f(x_3) \neq 0, \tag{7.7}$$

using the separation of variables. Namely, substituting (7.7) into (7.5) and dividing by nonvanishing product $f\psi$ we have

$$f\Delta\psi + f''\psi + k^2 f\psi = 0 \quad \text{or} \quad \frac{\Delta\psi}{\psi} + \frac{f''}{f} = -k^2, \tag{7.8}$$

which yields

$$\frac{\Delta\psi}{\psi} = -\lambda, \quad \frac{f''}{f} = \lambda - k^2 \tag{7.9}$$

with a certain constant λ. Thus ψ must solve the Dirichlet eigenvalue problem for the Laplace equation in cross-sectional domain Ω

$$\begin{aligned} \Delta\psi + \lambda\psi &= 0, \quad \mathbf{x}' \in \Omega, \\ \psi|_\Gamma &= 0. \end{aligned} \tag{7.10}$$

Denote by $\Lambda = \{\lambda_n\}$ and $\Psi = \{\psi_n\}$ the system of eigenvalues and eigenfunctions of this problem. A particular solution of (7.5) is

$$\Pi = \Pi_n(\mathbf{x}) = \psi_n(\mathbf{x}')f_n(x_3), \tag{7.11}$$

where f_n satisfies the equation

$$f_n'' + (k^2 - \lambda_n)f_n = 0. \tag{7.12}$$

The general solution of (7.12) is

$$f_n(x_3) = A_n e^{i\gamma_n x_3} + B_n e^{-i\gamma_n x_3}, \quad \gamma_n = \sqrt{k^2 - \lambda_n}. \tag{7.13}$$

The first and the second terms in (7.13) correspond, respectively, to the wave propagating in the positive or negative direction of the waveguide axis.

Considering the wave propagating in the positive direction set

$$f_n(x_3) = A_n e^{i\gamma_n x_3}. \tag{7.14}$$

As a result we obtain the solution

$$\Pi_n(\mathbf{x}', x_3) = A_n \psi_n(\mathbf{x}') e^{i\gamma_n x_3}. \tag{7.15}$$

We have $0 \le \lambda_1 \le \lambda_2 \le \ldots$; therefore, that are at most finitely many values of $\gamma_n = \sqrt{k^2 - \lambda_n}$ with $k^2 > \lambda_n$ that are real, while infinitely many of them, for $\gamma_n = i\sqrt{\lambda_n - k^2}$ ($i^2 = -1$) with $k^2 < \lambda_n$, are purely imaginary. Consequently, there are at most finitely many waves in the waveguide that propagate without attenuation (in the positive direction of x_3-axis) and infinitely many decay exponentially.

Looking for particular solutions of (7.1) in the form

$$\begin{aligned} \mathbf{H} &= \operatorname{grad}\operatorname{div}\mathbf{P} + k^2\mathbf{P}, \\ \mathbf{E} &= ik\operatorname{rot}\mathbf{P}, \end{aligned} \tag{7.16}$$

where the polarization potential $\mathbf{P} = [0,0,\Pi]$ has only one nonzero component $P_3 = \Pi$, it is easy to see that

$$E_3 = 0, \quad \mathbf{H} = [0,0,H_3], \quad \mathbf{E} = [E_1,E_2,0], \tag{7.17}$$

and this case is called TE-polarization or H-polarization. Substituting (7.16) into (7.1) yields the equations

$$\begin{aligned} \Delta_3\Pi + k^2\Pi &= 0 \quad \text{or} \quad \Delta\Pi + \frac{\partial^2\Pi}{\partial x_3^2} + k^2\Pi = 0, \\ \Delta_3 &= \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}. \end{aligned} \tag{7.18}$$

Condition (7.2) is satisfied if we assume that

$$\left.\frac{\partial \Pi}{\partial n}\right|_{\Sigma} = 0, \tag{7.19}$$

because the third components of $\mathbf{P}$ and the first two of $\mathbf{E}$ are tangential components that must vanish on the waveguide wall and they are coupled by the first relation (7.16). Repeating the above analysis we see that

$$\Pi = \Pi_n(\mathbf{x}) = A_n \psi_n(\mathbf{x}')e^{i\gamma_n x_3}, \tag{7.20}$$

where ψ_n solves the Neumann eigenvalue problem for the Laplace equation in cross-sectional domain Ω

$$\begin{aligned} \Delta\psi + \lambda\psi &= 0, \quad \mathbf{x}' \in \Omega, \\ \left.\frac{\partial \psi}{\partial n}\right|_{\Sigma} &= 0. \end{aligned} \tag{7.21}$$

(7.20) specifies the wave propagating in the positive direction of the waveguide axis. Denote by $\Lambda^H = \{\lambda_n^H\}$ and $\Psi^H = \{\psi_n^H\}$ the system of eigenvalues and eigenfunctions of this problem. We have $0 \le \lambda_1^H \le \lambda_2^H \le \ldots$; therefore, that are at most finitely many values of $\gamma_n^H = \sqrt{k^2 - \lambda_n^H}$ with $k^2 > \lambda_n^H$ that are real, while infinitely many of them, for $\gamma_n^H = i\sqrt{\lambda_n^H - k^2}$ with $k^2 < \lambda_n^H$ are purely imaginary. Consequently, there are at most finitely many waves in the waveguide that propagate without attenuation (in the positive direction of x_3-axis) and infinitely many decay exponentially.

The waves obtained from (7.3), (7.4) or (7.16), (7.17) are called, respectively, TM-waves or TE-waves.

7.2 Diffraction from a dielectric obstacle in a 2D-guide

Introduce the complex magnitude of the stationary electric and magnetic field, $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{H}(\mathbf{r},t)$, respectively, where $\mathbf{r} = (x,y,z)$, and consider the problem of diffraction of a TM wave (or mode)

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}(\mathbf{r})\exp(-i\omega t), \quad \mathbf{H}(\mathbf{r},t) = \mathbf{H}(\mathbf{r})\exp(-i\omega t), \tag{7.22}$$

$$\mathbf{E}(\mathbf{r}) = (E_x,0,0), \quad \mathbf{H}(\mathbf{r}) = \left(0, \frac{1}{i\omega\mu_0}\frac{\partial E_x}{\partial z}, -\frac{1}{i\omega\mu_0}\frac{\partial E_x}{\partial y}\right), \tag{7.23}$$

by a dielectric inclusion D in a parallel-plane waveguide $\mathscr{W} = \{\mathbf{r} : 0 < y < \pi, -\infty < x,z < \infty\}$.

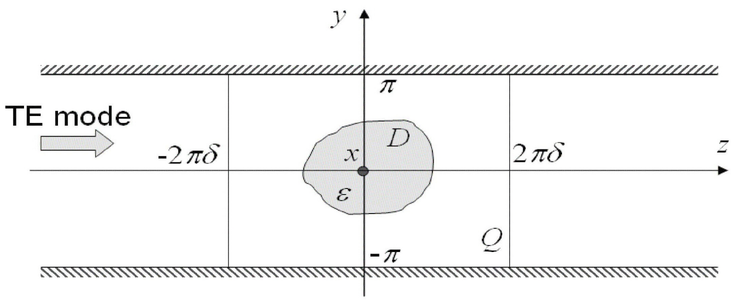

Figure 7.1: TE-mode diffraction by a dielectric inclusion in a parallel-plane waveguide

The total field $u(y,z) = E_x(y,z) = E_x^{inc}(y,z) + E_x^{scat}(y,z) = u^i(y,z) + u^s(y,z)$ of the diffraction by the D of the unit-magnitude TE wave with the only nonzero component is the solution to the BVP

$$[\Delta + \kappa^2\varepsilon(y,z)]u(y,z) = 0 \text{ in } S = \{(y,z) : 0 < y < \pi, -\infty < z < \infty\}, \quad u(\pm\pi,z) = 0, \tag{7.24}$$

$$u(y,z) = u^i(y,z) + u^s(y,z), \quad u^s(y,z) = \sum_{n=1}^{\infty} a_n^{\pm}\exp(i\Gamma_n z)\sin(ny), \tag{7.25}$$

where $\Delta = \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$ is the Laplace operator, superscripts $^+$ and $^-$ correspond, respectively, to the domains $z > 2\pi\delta$ and $z < -2\pi\delta$, $\omega = \kappa c$ is the dimensionless circular frequency, $\kappa = \omega/c = 2\pi/\lambda$ is the dimensionless frequency parameter (λ is the free-space wavelength), $c = (\varepsilon_0\mu_0)^{-1/2}$ is the speed of light in vacuum, and $\Gamma_n = (\kappa^2 - n^2)^{1/2}$ is the transverse wavenumber satisfying the conditions

$$Im\Gamma_n \geq 0, \quad \Gamma_n = i|\Gamma_n|, \quad |\Gamma_n| = Im\Gamma_n = (n^2 - \kappa^2)^{1/2}, \quad n > \kappa. \tag{7.26}$$

It is also assumed that the series in (7.25) converges absolutely and uniformly and allows for double differentiation with respect to y and z.

Note that $u^i(y,z)$ satisfies (7.24) in S, the boundary condition, and radiation condition (7.25) only in the positive direction, so that the electromagnetic field with the x-component $u^i(y,z)$ may be interpreted as a normal wave (a waveguide mode) coming from the domain $z < -2\pi\delta$.

7.3 Diffraction from a dielectric obstacle in a 3D-guide

Diffraction of electromagnetic waves by a dielectric body Q in a 3D tube (a waveguide) with cross section Ω (a 2D domain bounded by smooth curve Γ) parallel to the x_3-axis in the cartesian coordinate system is described by the solution to the inhomogeneous system of Maxwell equations

$$\begin{aligned} \mathrm{rot}\mathbf{H} &= -i\omega\hat{\varepsilon}\mathbf{E}+\mathbf{j}_E^0 \\ \mathrm{rot}\mathbf{E} &= i\omega\mu_0\mathbf{H}, \end{aligned} \tag{7.27}$$

$$\mathbf{E}_\tau|_{\partial P}=0, \quad \mathbf{H}_\nu|_{\partial P}=0, \tag{7.28}$$

admitting for $|x_3|>C$ and sufficiently large $C>0$ the representations (+ corresponds to $+\infty$ and $-$ to $-\infty$)

$$\begin{aligned} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} &= \sum_p R_p^{(\pm)} e^{-i\gamma_p^{(1)}|x_3|} \begin{pmatrix} \lambda_p^{(1)}\Pi_p e_3 - i\gamma_p^{(1)}\nabla_2\Pi_p \\ -i\omega\varepsilon_0(\nabla_2\Pi_p)\times e_3 \end{pmatrix} + \\ &+ \sum_p Q_p^{(\pm)} e^{-i\gamma_p^{(2)}|x_3|} \begin{pmatrix} i\omega\mu_0(\nabla_2\Psi_p)\times e_3 \\ \lambda_p^{(2)}\Psi_p e_3 - i\gamma_p^{(2)}\nabla_2\Psi_p \end{pmatrix}. \end{aligned} \tag{7.29}$$

Here, $\gamma_p^{(j)}=\sqrt{k_0^2-\lambda_p^{(j)}}$, $\mathrm{Im}\,\gamma_p^{(j)}<0$ or $\mathrm{Im}\,\gamma_p^{(j)}=0$, $k_0\gamma_p^{(j)}\geq 0$, and $\lambda_p^{(1)}$, $\Pi_p(x_1,x_2)$ and $\lambda_p^{(2)}$, $\Psi_p(x_1,x_2)$ ($k_0^2=\omega^2\varepsilon_0\mu_0$) are the complete system of eigenvalues and orthogonal and normalized in $L_2(\Pi)$ eigenfunctions of the two-dimensional Laplace operator $-\Delta$ in the rectangle $\Pi_{ab}=\{(x_1,x_2): 0<x_1<a, 0<x_2<b\}$ with the Dirichlet and the Neumann conditions, respectively; and $\nabla_2 \equiv e_1\partial/\partial x_1 + e_2\partial/\partial x_2$.

We assume that $\mathbf{E}^0$ and $\mathbf{H}^0$ are solutions of BVP under consideration in the absence of body Q, $\hat{\varepsilon}(x)=\varepsilon_0\hat{I}$, $x\in P$ ($\hat{I}$ is the identity tensor):

$$\begin{aligned} \mathrm{rot}\mathbf{H}^0 &= -i\omega\varepsilon_0\mathbf{E}^0+\mathbf{j}_E^0 \\ \mathrm{rot}\mathbf{E}^0 &= i\omega\mu_0\mathbf{H}^0, \end{aligned} \tag{7.30}$$

$$\mathbf{E}_\tau^0|_{\partial P}=0, \quad \mathbf{H}_\nu^0|_{\partial P}=0. \tag{7.31}$$

These solutions can be expressed in an analytical form in terms of $\mathbf{j}_\mathbf{E}^\mathbf{0}$ using Green's tensor of domain P. These solutions should not satisfy the conditions at infinity (7.29). For example, $\mathbf{E}^0$ and $\mathbf{H}^0$ can be TM- or TE-mode of this waveguide.

7.4 Problems

7.4.1 Problem

Determine explicit expressions for TM-waves in a waveguide of rectangular cross section $\Pi_{ab}=\{(x_1,x_2): 0<x_1<a, 0<x_2<b\}$.

7.4.2 Problem

Prove that

$$\mathbf{H}=[0,0,H_3], \quad \mathbf{E}=[E_1,E_2,0] \quad (E_3=0)$$

(formula (7.17)) if

$$\mathbf{H}=\mathrm{grad}\,\mathrm{div}\,\mathbf{P}+k^2\mathbf{P}, \quad \mathbf{E}=ik\mathrm{rot}\,\mathbf{P}, \quad \mathbf{P}=[0,0,\Pi].$$

7.4.3 Miniproject: energy of electromagnetic wave

The energy W_3 carried by the TM normal wave $\mathbf{E}, \mathbf{H}$ of index n propagating along x_3-axis in a waveguide with cross section Ω is determined according to

$$W_3 = \frac{\omega}{8\pi k} \int\int_{\Omega} [\mathbf{E}, \mathbf{H}^*]_{x_3} \, ds, \tag{7.32}$$

where * denotes complex conjugation. Prove that

$$W_3 = \frac{\omega}{8\pi} |A_n|^2 \gamma_n \lambda_n, \tag{7.33}$$

where λ_n is the nth eigenvalue of the Dirichlet boundary eigenvalue problem for the Laplace equation in Ω.
Hints: Use the solution to Problem 4.4.4 and formula (7.20).

7.4.4 Problem

The normal wave propagating along x_3-axis in a waveguide with cross section Ω that corresponds to the first (minimal) eigenvalue λ_1 of the Dirichlet boundary eigenvalue problem for the Laplace equation in Ω is often called *the fundamental TM mode* of the waveguide. Determine an explicit expression for the fundamental TM mode in a waveguide of rectangular cross section $\Omega = \Pi_{ab} = \{(x_1, x_2) : 0 < x_1 < a, 0 < x_2 < b\}$.

7.4.5 Problem

A solution to the homogeneous Maxwell equations satisfying homogeneous boundary conditions on the waveguide walls, independent of coordinate x, and with the dependence $e^{i\gamma z}$ on coordinate z is called a normal wave propagating along z-axis in a parallel plane (2D) waveguide $\mathscr{W}$. Determine an explicit expression for the fundamental TM mode (see (7.22), (7.23), (7.24)) in $\mathscr{W}$.

7.4.6 Problem

Prove that for $1 < \kappa < 2$ the scattered field component $u^s(y,z)$ in (7.25) can be represented asymptotically for $z > 2\pi\delta$ as

$$u^s(y,z) = U^1(y,z) + \exp(-|\Gamma_2|z) U^s(y,z), \quad U^1(y,z) = a_1^+ \exp(i\Gamma_1 z) \sin(y), \tag{7.34}$$

where $U^s(y,z)$ is a bounded and differentiable function in strip S satisfying $|U^s(y,z)| < U_0$, $(y,z) \in S$, with a certain constant U_0. How accurate can we determine $u^s(y,z)$ if we replace it with principal term $U^1(y,z)$?

8. Short introduction to vector differential calculus.

8.1 Gradient. Directional Derivative

Definition of the gradient. Vector function

$$\operatorname{grad} f = \nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

is called a gradient of (scalar) function $f(x,y,z)$.

Vector differential operator ∇ is defined by

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}.$$

Consider an example:

$$f(x,y,z) = 2x + yz - 3y^2; \quad \frac{\partial f}{\partial x} = 2, \quad \frac{\partial f}{\partial y} = z - 6y, \quad \frac{\partial f}{\partial z} = y,$$

$$\operatorname{grad} f = \nabla f = 2\mathbf{i} + (z-6y)\mathbf{j} + y\mathbf{k}.$$

Directional derivative. The directional derivative $D_{\mathbf{b}} f$ or $\dfrac{df}{ds}$ of a function f at a point P in the direction of a vector $\mathbf{b}$, $|\mathbf{b}| = 1$, is defined by

$$D_{\mathbf{b}} f = \lim_{s \to 0} \frac{f(Q) - f(P)}{s} \qquad (s = |Q - P|),$$

where Q is a variable point on the straight line C in the direction of $\mathbf{b}$.

In the Cartesian xyz-coordinates straight line C in parametric form is given by

$$\mathbf{r}(s) = x(s)\mathbf{i} + y(s)\mathbf{j} + z(s)\mathbf{k} = \mathbf{p_0} + s\mathbf{b}$$

where $\mathbf{b}$ is a unit vector and $\mathbf{p_0}$ the position vector of P. Applying the definition it is easy to check, using the chain rule, that $D_{\mathbf{b}}f = \frac{df}{ds}$ is the derivative of the function $f(x(s), y(s), z(s))$ with respect to s

$$D_{\mathbf{b}}f = \frac{df}{ds} = \frac{\partial f}{\partial x}x' + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial z}z',$$

$$x' = \frac{dx}{ds}, \quad y' = \frac{dy}{ds}, \quad z' = \frac{dz}{ds}.$$

Differentiation gives

$$\mathbf{r}'(s) = x'\mathbf{i} + y'\mathbf{j} + z'\mathbf{k} = \mathbf{b},$$

that is

$$D_{\mathbf{b}}f = \frac{df}{ds} = \mathbf{b} \cdot \operatorname{grad} f$$

($\mathbf{b}$ is a unit vector, $|\mathbf{b}| = 1$), or

$$D_{\mathbf{a}}f = \frac{df}{ds} = \frac{1}{|\mathbf{a}|}\mathbf{a} \cdot \operatorname{grad} f$$

where $\mathbf{a} \neq 0$ is a vector of any length).

Example 1 Gradient. Directional Derivative

Find the directional derivative of $f(x,y,z) = 2x^2 + 3y^2 + z^2$ at $P:(2,1,3)$ in the direction of $\mathbf{a} = \mathbf{i} - 2\mathbf{k} = [1,0,-2]$.

Solution.

$$f(x,y,z) = 2x^2 + 3y^2 + z^2; \quad \frac{\partial f}{\partial x} = 4x, \quad \frac{\partial f}{\partial y} = 6y, \quad \frac{\partial f}{\partial z} = 2z,$$

$$\operatorname{grad} f = 4x\mathbf{i} + 6y\mathbf{j} + 2z\mathbf{k}.$$

At the point $P:(2,1,3)$

$$\operatorname{grad} f = 8\mathbf{i} + 6\mathbf{j} + 6\mathbf{k} = [8,6,6].$$

Since $|\mathbf{a}| = |[1,0,-2]| = \sqrt{1+4} = \sqrt{5}$, we obtain

$$D_{\mathbf{a}}f = \frac{df}{ds} = \frac{1}{\sqrt{5}}(\mathbf{i} - 2\mathbf{k}) \cdot (8\mathbf{i} + 6\mathbf{j} + 6\mathbf{k}) =$$

$$\frac{1}{\sqrt{5}}[1,0,-2] \cdot [8,6,6] = \frac{1}{\sqrt{5}}(1 \cdot 8 + 0 \cdot 6 + (-2) \cdot 6) = -\frac{4}{\sqrt{5}} \approx -1.789.$$

Theorem 22 *grad f points in the direction of the maximum increase of f.*

Proof. From the definition of the scalar product we have

$$D_{\mathbf{b}}f = \mathbf{b} \cdot \operatorname{grad} f = |\mathbf{b}||\operatorname{grad} f| \cos\gamma = |\operatorname{grad} f| \cos\gamma \qquad (|\mathbf{b}| = 1).$$

where γ is the angle between $\mathbf{b}$ and $\operatorname{grad} f$. Directional derivative $D_{\mathbf{b}}f$ is maximum or minimum when $\cos\gamma = 1$, $\gamma = 0$, or, respectively $\cos\gamma = -1$, $\gamma = \pi$, that is if $\mathbf{b}$ is parallel to $\operatorname{grad} f$ or, respectively $-\operatorname{grad} f$. Thus, the following statement holds.

Theorem 23 Let $f(x,y,z) = f(P)$ be a differentiable function. Then directional derivative $D_{\mathbf{b}} f$ is (i) maximal in the direction

$$\mathbf{b} = \frac{\operatorname{grad} f}{|\operatorname{grad} f|}$$

and has the form

$$D_{\mathbf{b}} f = |\operatorname{grad} f|;$$

(ii) minimal in the direction

$$\mathbf{b} = -\frac{\operatorname{grad} f}{|\operatorname{grad} f|}$$

and has the form

$$D_{\mathbf{b}} f = -|\operatorname{grad} f|.$$

($\operatorname{grad} f \neq 0$).

8.2 Surface normal vector

Let S be a surface represented by

$$f(x,y,z) = c = const,$$

where f is a differentiable function.

Theorem 24 If $f(x,y,z) \in C^1)$ is a differentiable function and $\operatorname{grad} f \neq 0$ then $\operatorname{grad} f$ is a surface normal vector to the surface $f(x,y,z) = C$.

Proof. Let C be a curve on S through a point P of S. As a curve in space, C has a representation

$$\mathbf{r}(t) = \mathbf{v}(t) = [x(t), y(t), z(t)] = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}.$$

If C lies on surface S, the components of $\mathbf{r}(t)$ must satisfy $f(x,y,z) = C$, that is,

$$f(x(t), y(t), z(t)) = c.$$

A tangent vector to C is

$$\mathbf{r}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k};$$

the tangent vectors of all curves on S passing through P will generally form a plane called the *tangent plane* of S at P. The normal to this plane (a straight line through P perpendicular to the tangent plane) is called the *surface normal* to S at P. A vector in the direction of the surface normal is called a *surface normal vector* of S at P.

We can obtain such a vector by differentiating $f(x(t), y(t), z(t)) = c$ with respect to t. By the chain rule,

$$\frac{\partial f}{\partial x}x' + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial z}z' = \operatorname{grad} f \cdot \mathbf{r}'(t) = 0,$$

where

$$x' = \frac{dx}{dt}, \quad y' = \frac{dy}{dt}, \quad z' = \frac{dz}{dt}.$$

Hence $\operatorname{grad} f$ is orthogonal to all the vectors $\mathbf{r}'$ in the tangent plane, so that it is a normal vector of S at P.

Example 2 Gradient as Surface Normal Vector

Find a unit normal vector $\mathbf{n}$ of the cone of revolution $z^2 = 4(x^2 + y^2)$ at the point $P : (1, 0, 2)$.

Solution.

The cone is the level surface $z^2 = 4(x^2 + y^2)$, or

$$f(x,y,z) = 4x^2 + 4y^2 - z^2 = 0,$$

so that we have the equation of the cone as a level surface with $c = 0$. The partial derivatives are

$$\frac{\partial f}{\partial x} = 8x, \quad \frac{\partial f}{\partial y} = 8y, \quad \frac{\partial f}{\partial z} = -2z,$$

and the gradient is

$$\operatorname{grad} f = 8x\mathbf{i} + 8y\mathbf{j} - 2z\mathbf{k}.$$

At the point $P : (1,0,2)$

$$\operatorname{grad} f = 8\mathbf{i} - 4\mathbf{k} = [8, 0, -4].$$

We have $|\operatorname{grad} f| = \sqrt{64+16} = \sqrt{80}$. The unit normal vector of the cone at P is

$$\mathbf{n} = \frac{1}{|\operatorname{grad} f|}\operatorname{grad} f = \frac{1}{\sqrt{80}}(8\mathbf{i} - 4\mathbf{k}) = \frac{1}{4\sqrt{5}}4(2\mathbf{i} - \mathbf{k}) = \frac{2}{\sqrt{5}}\mathbf{i} - \frac{1}{\sqrt{5}}\mathbf{k}.$$

8.3 Gradient fields and potentials

Let a vector field be given by a vector function $\mathbf{p}$ which is the gradient of a scalar function, $\mathbf{p} = \operatorname{grad} f$. Function f is called a potential function or a potential of $\mathbf{p}$.

As an example, consider the vector function which describes the gravitational force (gravitational field)

$$\mathbf{p} = -c\left(\frac{x-x_0}{r^3}\mathbf{i} + \frac{y-y_0}{r^3}\mathbf{j} + \frac{z-z_0}{r^3}\mathbf{k}\right),$$

where

$$\mathbf{r} = [x-x_0, y-y_0, z-z_0] = (x-x_0)\mathbf{i} + (y-y_0)\mathbf{j} + (z-z_0)\mathbf{k}$$

and

$$r = |\mathbf{r}| = \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}.$$

We have a

$$\frac{\partial}{\partial x}\left(\frac{1}{r}\right) = \frac{-2(x-x_0)}{2[(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2]} = -\frac{x-x_0}{r^3}$$

and similarly

$$\frac{\partial}{\partial y}\left(\frac{1}{r}\right) = -\frac{y-y_0}{r^3}, \quad \frac{\partial}{\partial z}\left(\frac{1}{r}\right) = -\frac{z-z_0}{r^3}.$$

Thus $\mathbf{p}$ is the gradient of the scalar function

$$f(x,y,z) = \frac{c}{r} \qquad (r > 0):$$

$$\mathbf{p} = \operatorname{grad} f = \frac{\partial}{\partial x}\left(\frac{c}{r}\right)\mathbf{i} + \frac{\partial}{\partial y}\left(\frac{c}{r}\right)\mathbf{j} + \frac{\partial}{\partial z}\left(\frac{c}{r}\right)\mathbf{k}$$

According to the definition f is a scalar potential of the gravitational field.

Calculating the second partial derivatives with respect to x, y, z by the chain rule we obtain

$$\frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right) = -\frac{1}{r^3} + \frac{3(x-x_0)^2}{r^5},$$

$$\frac{\partial^2}{\partial y^2}\left(\frac{1}{r}\right) = -\frac{1}{r^3} + \frac{3(y-y_0)^2}{r^5},$$
$$\frac{\partial^2}{\partial z^2}\left(\frac{1}{r}\right) = -\frac{1}{r^3} + \frac{3(z-z_0)^2}{r^5}.$$

By adding the righthand and lefthand sides, one can show that potential f satisfies the Laplace equation

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

The differential operator of the second order

$$\Delta = \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

is called the Laplace operator (Laplacian).

8.4 Problems

8.4.1 Problem *8.8.1*

Find derivative $\frac{dw}{dt}$ of the function $w = \sqrt{x^2+y^2}$ where $x = e^{4t}$ and $y = e^{-4t}$.

8.4.2 Problem *8.9.1*

Find $\operatorname{grad} f$ of the function $f(x,y) = x^2 - y^2$ and its value and length at the point $P:(-1,3)$.

8.4.3 Problem *8.9.7*

Find the gradient $-\operatorname{grad} f$ for $f(x,y,z) = z/(x^2+y^2)$ and its value at the point $P:(0,1,2)$.

Recall that the directional derivative $D_{\mathbf{b}}f$ or $\frac{df}{ds}$ of a function f at a point P in the direction $\mathbf{b}$ is calculated as the scalar product

$$D_{\mathbf{b}}f = \frac{df}{ds} = \mathbf{b}\cdot\operatorname{grad} f$$

($\mathbf{b}$ is unit vector, $|\mathbf{b}| = 1$), or

$$D_{\mathbf{a}}f = \frac{1}{|\mathbf{a}|}\mathbf{a}\cdot\operatorname{grad} f$$

($\mathbf{a} \neq 0$ is an arbitrary vector).

8.5 Divergence and rotation of the vector field

Definition of divergence

Let

$$\mathbf{v}(x,y,z) = v_1(x,y,z)\mathbf{i} + v_2(x,y,z)\mathbf{j} + v_3(x,y,z)\mathbf{k}$$

be a differentiable vector function. The (scalar) function

$$\operatorname{div}\mathbf{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

is called the divergence of $\mathbf{v}$ or the divergence of the vector field defined by $\mathbf{v}$.

Define the vector differential operator ∇ by

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}.$$

Then we can write the divergence as the scalar product

$$\operatorname{div}\mathbf{v} = \nabla \cdot \mathbf{v} = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) \cdot (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}) =$$

$$\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}.$$

Consider an example:

$$\mathbf{v}(x,y,z) = 3xz\mathbf{i} + 2xy\mathbf{j} - yz^2\mathbf{k},$$

$$\frac{\partial v_1}{\partial x} = 3z, \quad \frac{\partial v_2}{\partial y} = 2x, \quad \frac{\partial v_3}{\partial z} = -2yz,$$

and

$$\operatorname{div}\mathbf{v} = 3z + 2x - 2yz.$$

If f is a twice differentiable function, then

$$\operatorname{grad} f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}.$$

and

$$\operatorname{div}\,(\operatorname{grad} f) = \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2},$$

so that

$$\operatorname{div}\,(\operatorname{grad} f) = \Delta f,$$

where Δ is the Laplace differential operator.

Example 3 Gravitational Force

The vector function

$$\mathbf{p} = -c\left(\frac{x-x_0}{r^3}\mathbf{i} + \frac{y-y_0}{r^3}\mathbf{j} + \frac{z-z_0}{r^3}\mathbf{k}\right),$$

where

$$\mathbf{r} = [x-x_0, y-y_0, z-z_0] = (x-x_0)\mathbf{i} + (y-y_0)\mathbf{j} + (z-z_0)\mathbf{k}$$

and

$$r = |\mathbf{r}| = \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2},$$

describes the gravitational force (gravitational field). We have

$$\frac{\partial}{\partial x}\left(\frac{1}{r}\right) = \frac{-2(x-x_0)}{2[(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2]} = -\frac{x-x_0}{r^3},$$

and similarly

$$\frac{\partial}{\partial y}\left(\frac{1}{r}\right) = -\frac{y-y_0}{r^3}, \quad \frac{\partial}{\partial z}\left(\frac{1}{r}\right) = -\frac{z-z_0}{r^3}.$$

Then $\mathbf{p}$ is the gradient of the function

$$f(x,y,z) = \frac{c}{r} \qquad (r > 0):$$

$$\mathbf{p} = \operatorname{grad} f = \frac{\partial}{\partial x}\left(\frac{c}{r}\right)\mathbf{i} + \frac{\partial}{\partial y}\left(\frac{c}{r}\right)\mathbf{j} + \frac{\partial}{\partial z}\left(\frac{c}{r}\right)\mathbf{k}$$

A vector field $\mathbf{p}$ is said to be a gradient of f if $\mathbf{p} = \operatorname{grad} f$; function f is called a scalar potential of $\mathbf{p}$. In the example above f is a scalar potential of the gravitational field.

Finding the second partial derivative using the chain rule with respect to x, y, z, we obtain

$$\frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right) = -\frac{1}{r^3} + \frac{3(x-x_0)^2}{r^5},$$

$$\frac{\partial^2}{\partial y^2}\left(\frac{1}{r}\right) = -\frac{1}{r^3} + \frac{3(y-y_0)^2}{r^5},$$

$$\frac{\partial^2}{\partial z^2}\left(\frac{1}{r}\right) = -\frac{1}{r^3} + \frac{3(z-z_0)^2}{r^5}.$$

By adding the righthand and lefthand sides, one can show that the potential f satisfies the Laplace equation

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0,$$

so that

$$\operatorname{div} \mathbf{p} = \operatorname{div}\,(\operatorname{grad} f) = \nabla^2 f = 0.$$

8.6 Rotation (curl) of a vector field

Definition of rotation.

Let x, y, z be a positive oriented Cartesian coordinate system and

$$\mathbf{v}(x,y,z) = v_1(x,y,z)\mathbf{i} + v_2(x,y,z)\mathbf{j} + v_3(x,y,z)\mathbf{k}$$

a differentiable vector function. Then the vector function

$$\operatorname{curl} \mathbf{v} = \nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix} =$$

$$\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)\mathbf{k}$$

is called rotation (or curl) of vector field $\mathbf{v}$.

Example 4 Curl of a the vector field

Let x, y, z be a positive oriented Cartesian coordinate system. Consider the vector field

$$\mathbf{v}(x,y,z) = yz\mathbf{i} + 3zx\mathbf{j} + z\mathbf{k}.$$

The curl of $\mathbf{v}$ is calculated according to

$$\operatorname{curl} \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & 3xz & z \end{vmatrix} =$$

$$\left(\frac{\partial z}{\partial y} - \frac{\partial(3xz)}{\partial z}\right)\mathbf{i} + \left(\frac{\partial(yz)}{\partial z} - \frac{\partial z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial(3xz)}{\partial x} - \frac{\partial(yz)}{\partial y}\right)\mathbf{k} = -3x\mathbf{i} + y\mathbf{j} + 2z\mathbf{k}.$$

8.7 Important vector differential identities

Theorem 25 For any twice continuously differentiable scalar function f,

$$\operatorname{curl}(\operatorname{grad} f) = \mathbf{0}. \tag{8.1}$$

The potential (or conservative) field is called *rotation-free*.

Proof.

$$\operatorname{curl}(\operatorname{grad} f) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_x & f_y & f_z \end{vmatrix} =$$

$$\left(\frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial f_x}{\partial z} - \frac{\partial f_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}\right)\mathbf{k} = (f_{zy} - f_{yz})\mathbf{i} + (f_{xz} - f_{zx})\mathbf{j} + (f_{yx} - f_{xy})\mathbf{k} = \mathbf{0}.$$

Theorem 26 For any twice continuously differentiable vector function $\mathbf{v}$,

$$\operatorname{div}(\operatorname{curl} \mathbf{v}) = 0. \tag{8.2}$$

The field of rotation is called *divergence-free*.
Proof.

$$\operatorname{div}(\operatorname{curl} \mathbf{v}) = \frac{\partial}{\partial x}\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right) =$$

$$(v_{3yx} - v_{2zx}) + (v_{1zy} - v_{3xy}) + (v_{2xz} - v_{1yz}) = 0.$$

More vector differential identities:

$$\begin{aligned} \nabla(\phi\psi) &= \psi\nabla\phi + \phi\nabla\psi. \\ \nabla\cdot(\phi\mathbf{F}) &= \operatorname{div}(\phi\mathbf{F}) = \nabla\phi\cdot\mathbf{F} + \phi\nabla\cdot\mathbf{F}. \\ \nabla\cdot(\mathbf{F}\times\mathbf{G}) &= \nabla\times\mathbf{F}\cdot\mathbf{G} - \mathbf{F}\cdot\nabla\times\mathbf{G}. \\ \nabla\times(\nabla\times\mathbf{F}) &= \nabla(\nabla\cdot\mathbf{F}) - \nabla^2\mathbf{F}. \end{aligned} \tag{8.3}$$

8.8 Problems

8.8.1 Problem *8.10.1*

Determine the divergence of

$$\mathbf{v}(x,y,z) = v_1(x,y,z)\mathbf{i} + v_2(x,y,z)\mathbf{j} + v_3(x,y,z)\mathbf{k} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

8.8.2 Problem *8.10.2*

Determine the divergence of

$$\mathbf{v}(x,y,z) = v_1(x,y,z)\mathbf{i} + v_2(x,y,z)\mathbf{j} + v_3(x,y,z)\mathbf{k} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}.$$

8.8.3 Problem *8.10.14*

Determine $\Delta f = \nabla^2 f$ of the function

$$f(x,y) = (x-y)/(x+y).$$

8.8.4 **Problem** *8.11.3*

Find curl of the vector field

$$\mathbf{v} = \frac{1}{2}(x^2 + y^2 + z^2)(\mathbf{i} + \mathbf{j} + \mathbf{k}).$$

8.8.5 **Problem** *8.11.14*

Show that

$$\operatorname{div}(\mathbf{u} \times \mathbf{v}) = \mathbf{v} \cdot \operatorname{curl} \mathbf{u} - \mathbf{u} \cdot \operatorname{curl} \mathbf{v}.$$

8.8.6 **Problem** *8.11.15*

Find curl $f\mathbf{u}$ of

$$\mathbf{u} = y\mathbf{i} + z\mathbf{j} + x\mathbf{k}$$

where $f = xyz$.

9. Line integrals

9.1 Curves in a parametric form and line integrals

Let xyz be a Cartesian coordinate system in space. We write a spatial curve C using a parametric representation

$$\mathbf{r}(t) = [x(t), y(t), z(t)] = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} \qquad (t \in I), \tag{9.1}$$

where variable t is a parameter.

As far as a line integral over a spatial curve C is concerned, C is called the path of integration. The path of integration with spatial endpoints A to B goes from A to B (has a certain direction) so that $A := \mathbf{r}(a)$ is its initial point and $B := \mathbf{r}(b)$ is its terminal point. C is now oriented. The direction from A to B, in which t increases is called the positive direction on C. Points A and B may coincide, then C is called a closed path.

Example 5 Elliptical arc

The vector function

$$\mathbf{r}(t) = [a\cos t, b\sin t], \quad t:\ \pi \to 0$$

defines an oriented elliptical arc (on the xy-plane). The corresponding parameter interval has the endpoints $a = \pi$ and $b = 0$. With such an orientation,

$$P(\pi) = (a\cos\pi, b\sin\pi) = (-a, 0)$$

is the initial point and

$$P(0) = (a\cos 0, b\sin 0) = (a, 0)$$

is the endpoint. So the elliptical arc has become an oriented curve.

9.2 Line integrals

Definition of line integral. If C is an oriented curve in a parametric form

$$P = P(t) \quad (x = x(t), \quad y = y(t), \quad z = z(t)) \quad t \in I = (t_0, t_1), \quad t: \ t_0 \to t_1, \tag{9.2}$$

and $f(P)$ and $g(P)$ are real (or complex) function defined on C, the *line integral of a scalar function* is defined as

$$\int_C f(P)dg(P) = \int_{t=t_0}^{t=t_1} f(P(t))dg(P(t)), \tag{9.3}$$

(if the right-hand side in the equality specifying the integral exists).

A *line integral* of a vector function $\mathbf{F}(\mathbf{r})$ over a curve C is defined by

$$\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \frac{d\mathbf{r}}{dt} dt,$$

or componentwise

$$\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_C (F_1 dx + F_2 dy + F_3 dz) = \int_a^b (F_1 x' + F_2 y' + F_3 z')dt \quad (' = d/dt).$$

Example 6 A line integral in the plane

Find the value of the line integral when $\mathbf{F}(\mathbf{r}) = [-y, -xy]$ and C is a circular arc from $(1,0)$ to $(0,1)$.

Solution. We may represent C by

$$\mathbf{r}(t) = [\cos t, \sin t] = \cos t\mathbf{i} + \sin t\mathbf{j}, \tag{9.4}$$

and

$$\mathbf{r}(t) = [\cos t, \sin t], \quad t: \ 0 \to \pi/2.$$

The parameter interval is $I = (t_0, t_1)$ with the initial point $t_0 = 0$ and endpoint $t_1 = \pi/2$. In such an orientation,

$$P(0) = (\cos 0, b \sin 0) = (1, 0)$$

is the initial point and

$$P(\pi/2) = (a \cos \pi/2, b \sin \pi/2) = (0, 1)$$

is the endpoint.

We have $x = \cos t, \ y = \sin t$ and can write vector function $\mathbf{F}(\mathbf{r})$ on the unit circle

$$\mathbf{F}(\mathbf{r}(t)) = -y(t)\mathbf{i} - x(t)y(t)\mathbf{j} = [-\sin t, -\cos t \sin t] = -\sin t\mathbf{i} - \cos t \sin t\mathbf{j}.$$

Determine

$$\mathbf{r}'(t) = -\sin t\mathbf{i} + \cos t\mathbf{j}$$

and calculate the line integral:

$$\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_0^{\pi/2} (-\sin t\mathbf{i} - \cos t \sin t\mathbf{j}) \cdot (-\sin t\mathbf{i} + \cos t\mathbf{j})dt =$$

$$\int_0^{\pi/2} (\sin^2 t - \cos^2 t \sin t)dt = \int_0^{\pi/2} [(1/2)(1 - \cos 2t) - \cos^2 t \sin t]dt =$$

$$(1/2) \int_0^{\pi/2} [(1 - \cos 2t)dt + \int_0^{\pi/2} \cos^2 t d \cos t = \frac{\pi}{4} - \frac{1}{3}.$$

Example 7 Line integral depends on the form of the curve

Find the line integral for $\mathbf{F}(\mathbf{r}) = [5z, xy, x^2 z]$ when curves C_1 and C_2 have the same initial point $A : (0,0,0)$ and endpoint $B : (1,1,1)$, C_1 is an interval of the straight line

$$\mathbf{r}_1(t) = [t,t,t] = t\mathbf{i} + t\mathbf{j} + t\mathbf{k}, \quad 0 \le t \le 1,$$

and C_2 is a parabola

$$\mathbf{r}_2(t) = [t,t,t^2] = t\mathbf{i} + t\mathbf{j} + t^2\mathbf{k}, \quad 0 \le t \le 1.$$

Solution. We have

$$\mathbf{F}(\mathbf{r}_1(t)) = 5t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}, \quad \mathbf{F}(\mathbf{r}_2(t)) = 5t^2\mathbf{i} + t^2\mathbf{j} + t^4\mathbf{k},$$

$$\mathbf{r}_1'(t) = \mathbf{i} + \mathbf{j} + \mathbf{j}, \quad \mathbf{r}_2'(t) = \mathbf{i} + \mathbf{j} + 2t\mathbf{j}.$$

Then we can calculate the line integral over C_1

$$\int_{C_1} \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_0^1 \mathbf{F}(\mathbf{r}_1(t)) \cdot \mathbf{r}_1'(t) dt =$$

$$\int_0^1 (5t + t^2 + t^3) dt = \frac{5}{2} + \frac{1}{3} + \frac{1}{4} = \frac{37}{12}.$$

The line integral over C_2 is

$$\int_{C_2} \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_0^1 \mathbf{F}(\mathbf{r}_2(t)) \cdot \mathbf{r}_2'(t) dt =$$

$$\int_0^1 (5t^2 + t^2 + 2t^5) dt = \frac{5}{3} + \frac{1}{3} + \frac{2}{6} = \frac{28}{12}.$$

Thus we have got two different values.

Theorem 27 The line integral

$$\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_C (F_1 dx + F_2 dy + F_3 dz),$$

where F_1, F_2, F_3 are continuous functions on a domain D in space, is path independent in D, if and only if $\mathbf{F} = [F_1, F_2, F_3]$ is the gradient of a function $f = f(x,y,z)$ in D :

$$\mathbf{F} = \operatorname{grad} f;$$

with the components

$$F_1 = \frac{\partial f}{\partial x}, \quad F_2 = \frac{\partial f}{\partial y}, \quad F_3 = \frac{\partial f}{\partial z}.$$

If $\mathbf{F}$ is the gradient field and f is a scalar potential of $\mathbf{F}$ then the line integral

$$\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = f(B) - f(A),$$

where A is the initial point and B the endpoint of C.

Example 8 Path independence

Show that the integral

$$\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_C (2xdx + 2ydy + 4zdz)$$

is path independent in any domain in space and find its value if integration is performed from $A : (0,0,0)$ to $B : (2,2,2)$.

Solution. We have

$$\mathbf{F} = [2x, 2y, 4z] = 2x\mathbf{i} + 2y\mathbf{j} + 4z\mathbf{k} = \operatorname{grad} f,$$

and it is easy to check that

$$f(x,y,z) = x^2 + y^2 + 2z^2.$$

According to Theorem 27, the line integral is path independent in any domain in space. To find its value, we choose the convenient straight path

$$\mathbf{r}(t) = [t,t,t] = t(\mathbf{i} + \mathbf{j} + \mathbf{k}), \quad 0 \le t \le 2.$$

Let $A : (0,0,0)$, $t = 0$, be the initial point and $B : (2,2,2)$, $t = 2$ the endpoint. Then we get

$$\mathbf{r}'(t) = \mathbf{i} + \mathbf{j} + \mathbf{j}.$$

$$\mathbf{F}(\mathbf{r}) \cdot \mathbf{r}' = 2t + 2t + 4t = 8t$$

and

$$\int_C (2xdx + 2ydy + 4zdz) = \int_0^2 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)dt = \int_0^2 8tdt = 16.$$

According to Theorem 27,

$$\int_C \mathbf{F}(\mathbf{r})d\mathbf{r} = f(2,2,2) - f(0,0,0) = 4 + 4 + 2 \cdot 4 - 0 = 16.$$

Example 9 Path independence. Determination of a potential

Evaluate the integral

$$I = \int_C (3x^2dx + 2yzdy + y^2dz)$$

from $A : (0,1,2)$ to $B : (1,-1,7)$:

Solution. If $\mathbf{F}$ has a potential f, then

$$\mathbf{F} = \operatorname{grad} f : \quad \frac{\partial f}{\partial x} = F_1 = 3x^2, \ \frac{\partial f}{\partial y} = F_2 = 2yz, \ \frac{\partial f}{\partial z} = F_3 = y^2.$$

Performing integration, we obtain

$$f = x^3 + g(y,z), \quad f_y = g_y = 2yz, \ g = y^2z + h(z),$$

$$f_z = y^2 + h' = y^2, \quad h' = 0, \ h = 0.$$

This gives

$$f(x,y,z) = x^3 + y^2z$$

and

$$I = f(1,-1,7) - f(0,1,2) = 1 + 7 - (0 + 2) = 6.$$

Theorem 28 The line integral

$$\int_C \mathbf{F}(\mathbf{r})\cdot d\mathbf{r} = \int_C (F_1 dx + F_2 dy + F_3 dz)$$

where F_1, F_2, F_3 are continuous functions on a domain D in space is path independent in D if and only if

$$\int_C \mathbf{F}(\mathbf{r})\cdot d\mathbf{r} = 0$$

along every closed path C in D.

The differential form

$$F_1 dx + F_2 dy + F_3 dz$$

is called exact in a domain D in space if it is the differential

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

of a differentiable function $f(x,y,z)$ everywhere in D:

$$F_1 dx + F_2 dy + F_3 dz = df,$$

where

$$F_1 = \frac{\partial f}{\partial x}, \quad F_2 = \frac{\partial f}{\partial y}, \quad F_3 = \frac{\partial f}{\partial z}.$$

Theorem 29 Consider the line integral

$$\int_C \mathbf{F}(\mathbf{r})\cdot d\mathbf{r} = \int_C (F_1 dx + F_2 dy + F_3 dz),$$

where F_1, F_2, F_3 continuous and have continuous first partial derivatives in a domain D in space. If the differential form $F_1 dx + F_2 dy + F_3 dz$ is exact in D then

$$\operatorname{curl}\mathbf{F} = \mathbf{0} \quad \text{in} \quad D;$$

in components,

$$\frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z}, \quad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}, \quad \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}.$$

If

$$\operatorname{curl}\mathbf{F} = \mathbf{0} \quad \text{in} \quad D$$

and *D is a simply connected, then the line integral is path independent in D.*

Suppose that F_1, F_2, F_3 are differentiable functions of three variables x, y, z in any open simply connected domain D. Then, by Theorem 29, the following conditions are equivalent:

(i.3) Differential form $F_1 dx + F_2 dy + F_3 dz$ is exact.

(ii.3)

$$\frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z}, \quad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}, \quad \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}.$$

(iii.3) $\int_C (F_1 dx + F_2 dy + F_3 dz) = 0$ for any closed curve C in D.

(iv.3) $\int_L (F_1 dx + F_2 dy + F_3 dz)$ depends only on the initial point and endpoint of the curve L in D (is path independent in D).

The same statements are valid in the two-dimensional case. Suppose that $F_1$and F_2 is differentiable functions of two variables x, y in an open, simply connected domain D. Then, by Theorem 29, the following conditions are equivalent:

(i.2) Differential form $F_1dx + F_2dy$ is exact.

(ii.2)

$$\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}.$$

(iii.2) $\int_C F_1dx + F_2dy = 0$ for any closed curve C in D.

(iv.2) $\int_L F_1dx + F_2dy$ depends only on the initial point and endpoint of the curve L in D (is path independent in D).

9.3 Problems

9.3.1 Problem *9.1.1*

Calculate

$$\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \frac{d\mathbf{r}}{dt} dt$$

where $\mathbf{F} = [y^2, -x^2]$ and C is an interval of the straight line from $(0,0)$ to $(1,4)$.

9.3.2 Problem *9.1.5*

Calculate

$$\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r},$$

where $\mathbf{F} = [(x-y)^2, (y-x)^2]$ and $C : xy = 1,\ 1 \le x \le 4$.

9.3.3 Problem *9.2.2*

Calculate the line integral

$$I = \int_C e^x(\cos y dx - \sin y dy)$$

from $A : (0, \pi)$ to $B : (3, \pi/2)$.

10. Green's Theorem in the Plane

Let C be a closed curve in xy-plane that does not intersect itself and makes just one turn in the positive direction (counterclockwise). Let $F_1(x,y)$ and $F_2(x,y)$ be functions that are continuous and have continuous partial derivatives $\dfrac{\partial F_1}{\partial y}$ and $\dfrac{\partial F_2}{\partial x}$ everywhere in some domain R enclosed by C. Then

$$\int_R\int \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy = \int_C (F_1 dx + F_2 dy).$$

Here we integrate along the entire boundary C of R so that R is on the left as we advance in the direction of integration.

One can write Green's formula with the help of curl

$$\int_R\int (\operatorname{curl}\mathbf{F})\cdot\mathbf{k}\,dxdy = \int_C \mathbf{F}\cdot d\mathbf{r}.$$

Example 10 Green's formula in plane

Verify Green's formula for $F_1 = y^2 - 7y$, $F_2 = 2xy + 2x$ and C being a circle $R: x^2 + y^2 = 1$.

Solution. Calculate a double integral

$$\int_R\int\left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)dxdy=\int_R\int[(2y+2)-(2y-7)]dxdy=9\int_R\int dxdy=9\pi.$$

Calculate the corresponding line integral. Circle C in the parametric form is given by

$$\mathbf{r}(t)=[\cos t,\sin t]=\cos t\mathbf{i}+\sin t\mathbf{j}.$$
$$\mathbf{r}'(t)=-\sin t\mathbf{i}+\cos t\mathbf{j}.$$

On C

$$F_1=y^2-7y=\sin^2 t-7\sin t,\quad F_2=2xy+2x=2\cos t\sin t+2\cos t,$$

and we get that the line integral in Green's formula is equal to the double integral:

$$\int_C\mathbf{F}(\mathbf{r})\cdot d\mathbf{r}=\int_0^{2\pi}[(\sin^2 t-7\sin t)(-\sin t)+(2\cos t\sin t+2\cos t)\cos t]dt=0+7\pi+0+2\pi=9\pi.$$

Example 11 Suppose that $w=w(x,y)$ is a differentiable function and

$$F_1=-\frac{\partial w}{\partial y},\quad F_2=\frac{\partial w}{\partial x}.$$

Then

$$\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}=\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}=\nabla^2 w.$$

The line integral

$$\int_C(F_1dx+F_2dy)=\int_C(F_1x'+F_2y')ds=\int_C(-\frac{\partial w}{\partial y}x'+\frac{\partial w}{\partial x}y')ds=\int_C(\operatorname{grad}w)\cdot\mathbf{n}ds,$$

where $\mathbf{n}$ is a unit normal vector to C because $\mathbf{n}$ is the unit tangent vector of C,

$$\mathbf{r}'(s)=x'\mathbf{i}+y'\mathbf{j}:\quad \mathbf{r}'(s)\cdot\mathbf{n}=0.$$

By the definition, $\operatorname{grad}w\cdot\mathbf{n}$ is the normal derivative of w in the direction of $\mathbf{n}$ and is denoted by $\frac{\partial w}{\partial n}$, Then the double integral of the Laplace operator applied to w, $\Delta w=\nabla^2 w$, is

$$\int_R\int\nabla^2 wdxdy=\int_C\frac{\partial w}{\partial n}ds.$$

10.1 Problems

10.1.1 Problem *9.4.1*

Evaluate the integral

$$I=\int_C\mathbf{F}(\mathbf{r})\cdot d\mathbf{r}=\int_a^b\mathbf{F}(\mathbf{r}(t))\cdot\frac{d\mathbf{r}}{dt}dt$$

counterclockwise along boundary C of region R by Green's theorem, where $\mathbf{F}=[x^2e^y,y^2e^x]$ and R is the rectangle with vertices $(0,0)$, $(2,0)$, $(2,3)$ and $(0,3)$.

10.1.2 Problem *9.4.3*

Evaluate the integral

$$I=\int_C\mathbf{F}(\mathbf{r})\cdot d\mathbf{r},$$

counterclockwise around boundary C of region R by Green's theorem, where $\mathbf{F}=[y,-x]$ and C is $x^2+y^2=1/4$.

11. Surfaces and integrals

11.1 Surfaces in a parametric form

Let xyz be a Cartesian coordinate system in space. A surface S in a parametric form is given by three equations

$$x = x(u,v), \quad y = y(u,v), \quad z = z(u,v), \qquad (u,v) \in D, \tag{11.1}$$

or

$$\mathbf{r}(u,v) = [x(u,v), y(u,v), z(u,v)] = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k} \quad [(u,v) \in D], \tag{11.2}$$

where variables u, v are called parameters. Domain D is located in the uv-plane and is called the parameter domain. (11.2) can be written as

$$\mathbf{r} = \mathbf{r}(u,v) \qquad [(u,v) \in D], \tag{11.3}$$

where $\mathbf{r}(u,v)$ is the position vector for a point on S,

$$\mathbf{r}(u,v) = \overrightarrow{OP(u,v)}.$$

A surface S may be given explicitly with respect to any of the coordinate pairs by one of the equations

$$z = f(x,y),$$

$$x = g(y,z)$$

or

$$y = h(x,z);$$

the parameter equations may be replaced by

$$x = u, \quad y = v, \quad z = f(x,y),$$

etc., and the corresponding vector parameter equation is

$$\mathbf{r} = [u, v, f(u,v)] \quad [(u,v) \in D]. \tag{11.4}$$

The parameter domain R is the projection of S on the xy- (yz-, xz-) plane.

A surface may be given implicitly by the equation

$$g(x,y,z) = 0;$$

e.g.,

$$x^2 + y^2 + z^2 = a^2, \quad z \geq 0,$$

or

$$z = +\sqrt{a^2 - x^2 - y^2}$$

gives the hemisphere of radius a and origin O.

Example 12 Parametric representation of a cylinder

The circular cylinder

$$x^2 + y^2 = a^2, \quad -\infty \leq z \leq \infty, \tag{11.5}$$

has radius a, height 2, and Oz as its axis. A parametric representation is

$$\mathbf{r}(u,v) = [a\cos u, a\sin u, v] = a\cos u\mathbf{i} + a\sin u\mathbf{j} + v\mathbf{k},$$

$$u, v \text{ i rektangel } R: \ 0 \leq u \leq 2\pi, \quad -\infty \leq v \leq \infty.$$

The components of $\mathbf{r}(u,v)$ are

$$x = a\cos u, \quad y = a\sin u, \quad z = v. \tag{11.6}$$

Note that each point x, y, z defined by (11.6) satisfies the cylinder equation (11.5), and conversely each point x, y, z on the surface of the cylinder satisfying (11.5)] can be written using parametrization (11.6) since

$$x^2 + y^2 = a^2\cos^2 u + a^2\sin^2 u = a^2(\cos^2 u + \sin^2 u) = a^2.$$

An equation

$$x^2 + y^2 = a^2, \quad -1 \leq z \leq 1$$

definies a cylindrical surface which has radius a, height 2, and Oz as its axis. A parametric representation is

$$\mathbf{r}(u,v) = [a\cos u, a\sin u, v] = a\cos u\mathbf{i} + a\sin u\mathbf{j} + v\mathbf{k},$$

$$u, v \text{ in rectangle } R: \ 0 \leq u \leq 2\pi, \ -1 \leq v \leq 1.$$

$\mathbf{r}(u,v)$'s components are

$$x = a\cos u, \quad y = a\sin u, \quad z = v.$$

Example 13 Parametric representation of a sphere

A sphere $x^2 + y^2 + z^2 = a^2$ can be represented by

$$\mathbf{r}(u,v) = a\cos v\cos u\mathbf{i} + a\cos v\sin u\mathbf{j} + a\sin v\mathbf{k},$$

$$u, v \text{ in rectangle } R: \ 0 \leq u \leq 2\pi, \ -\pi/2 \leq v \leq \pi/2.$$

The components of $\mathbf{r}(u,v)$ are given by

$$x = a\cos v\cos u, \quad y = a\cos v\sin u, \quad z = a\sin v.$$

Make use of the spherical coordinates

$$x = r\cos v\cos u, \quad y = r\cos v\sin u, \quad z = r\sin v,$$

where r is the distance to the origin and u and v are two angles. One also uses spherical coordinates in the form

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta, \quad r \geq 0,\ 0 \leq \theta \leq \pi,\ -\pi \leq \phi \leq \pi.$$

Example 14 Parametric representation of a cone

A circular cone $z = +\sqrt{x^2+y^2}$, $0 \leq z \leq H$ can be represented by

$$\mathbf{r}(u,v) = u\cos v\mathbf{i} + u\sin v\mathbf{j} + u\mathbf{k}, \quad u,v \text{ in rectangle } R: 0 \leq v \leq 2\pi,\ 0 \leq u \leq H.$$

The components of $\mathbf{r}(u,v)$ are given by

$$x = u\cos v, \quad y = u\sin v, \quad z = u.$$

observe that $x^2+y^2 = z^2$.

11.2 Tangent Plane and Surface Normal

Let C be a curve on a surface S given by parametric equations

$$u = u(t), \quad v = v(t)$$

and $\tilde{\mathbf{r}}(u(t),v(t))$ is the position vector of point P lying on C. According to the chain rule, we get a tangent vector tocurve C

$$\tilde{\mathbf{r}}'(t) = \frac{d\tilde{\mathbf{r}}}{dt} = \frac{\partial\mathbf{r}}{\partial u}u' + \frac{\partial\mathbf{r}}{\partial v}v'.$$

Hence the partial derivatives $\mathbf{r}_u$ and $\mathbf{r}_v$ at P are tangential to S at P. We assume that they are linearly independent, which geometrically means that the curves $u = const$ and $v = const$ on S intersect at P at a nonzero angle. Then $\mathbf{r}_u$ and $\mathbf{r}_v$ span the tangent plane of S at P. Hence their cross product gives a normal vector N of S at P:

$$\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v \neq \mathbf{0}.$$

The corresponding unit normal vector $\mathbf{n}$ of S at P is

$$\mathbf{n} = \frac{1}{|\mathbf{N}|}\mathbf{N} = \frac{1}{|\mathbf{r}_u \times \mathbf{r}_v|}\mathbf{r}_u \times \mathbf{r}_v.$$

If S is represented by an implicit equation

$$g(x,y,z) = 0,$$

then

$$\mathbf{n} = \frac{1}{|\operatorname{grad} g|}\operatorname{grad} g.$$

Example 15 Unit normal vector of a sphere

$g(x,y,z) = x^2 + y^2 + z^2 - a^2 = 0$:

$$\mathbf{n} = \frac{1}{|\operatorname{grad} g|} \operatorname{grad} g = \frac{1}{a}\mathbf{r} = \left[\frac{x}{a}, \frac{y}{a}, \frac{z}{a}\right] = \frac{x}{a}\mathbf{i} + \frac{y}{a}\mathbf{j} + \frac{z}{a}\mathbf{k}.$$

Example 16 Unit normal vector of a cone

$g(x,y,z) = -z + \sqrt{x^2 + y^2} = 0$:

$$\mathbf{n} = \frac{1}{|\operatorname{grad} g|} \operatorname{grad} g = \frac{1}{\sqrt{2}}\left[\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, -1\right] =$$
$$\frac{x}{\sqrt{x^2+y^2}}\mathbf{i} + \frac{y}{\sqrt{x^2+y^2}}\mathbf{j} - \mathbf{k}.$$

11.3 Surface Integrals

To define a surface integral, we take a surface S given by a parametric representation

$$\mathbf{r}(u,v) = [x(u,v), y(u,v), z(u,v)] = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}, \quad u,v \in R,$$

the normal vector

$$\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v \neq \mathbf{0},$$

and unit normal vector

$$\mathbf{n} = \frac{1}{|\mathbf{N}|}\mathbf{N}.$$

A **surface integral** of a vector function $\mathbf{F}(\mathbf{r})$ over a surface S is defined as

$$\int_S\int \mathbf{F}\cdot\mathbf{n}dA = \int_R\int \mathbf{F}(\mathbf{r}(u,v))\cdot\mathbf{N}(u,v)dudv. \tag{11.7}$$

Note that

$$\mathbf{n}dA = \mathbf{n}|\mathbf{N}|dudv = |\mathbf{N}|dudv,$$

and we assume that the parameters u,v belongs to a region R in the u,v-plane.

Write the equivalent expression componentwise using directional cosine:

$$\mathbf{F} = [F_1, F_2, F_3] = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k},$$

$$\mathbf{n} = [\cos\alpha, \cos\beta, \cos\gamma] = \cos\alpha\mathbf{i} + \cos\beta\mathbf{j} + \cos\gamma\mathbf{k},$$
$$\mathbf{N} = [N_1, N_2, N_3] = N_1\mathbf{i} + N_2\mathbf{j} + N_3\mathbf{k},$$

and

$$\int_S\int \mathbf{F}\cdot\mathbf{n}dA = \int_S\int (F_1\cos\alpha + F_2\cos\beta + F_3\cos\gamma)dA =$$
$$\int_S\int (F_1N_1 + F_2N_2 + F_3N_3)dudv.$$

Flux through a surface

Surface integral (11.7) can be interpreted as $\mathbf{F}$'s flux through a surface S.

Example 17 Determine the flux of water through the parabolic cylinder

$$S: y = x^2, \ 0 \le x \le 2, \ 0 \le z \le 3$$

if the velocity vector is $\mathbf{v} = \mathbf{F} = [3z^2, 6, 6xz]$.

Solution. A parametric representation of S is

$$\mathbf{r}(u,v) = [u, u^2, v] = u\mathbf{i} + u^2\mathbf{j} + v\mathbf{k}, \qquad 0 \le u \le 2, \ 0 \le v \le 3$$

(setting $x = u$ and $z = v$, we have $y = x^2 = u^2$).

Partial derivatives (vector functions) of $\mathbf{r}_u$ and $\mathbf{r}_v$,

$$\mathbf{r}_u = [1, 2u, 0], \quad \mathbf{r}_v = [0, 0, 1],$$

are tangent vectors to surface S at a point $P \in S$ spanning the tangent plane to S at point P. The vector (cross) product

$$\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v \neq \mathbf{0}$$

is a normal vector to S at point P (the cross product is perpendicular to the tangential plane). We have

$$\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2u & 0 \\ 0 & 0 & 1 \end{vmatrix} = 2u\mathbf{i} - \mathbf{j} = [2u, -1, 0].$$

The corresponding unit normal vector

$$\mathbf{n} = \frac{1}{|\mathbf{N}|}\mathbf{N} = \frac{1}{\sqrt{1+4u^2}}(2u\mathbf{i} - \mathbf{j}).$$

On surface S,

$$\mathbf{F}(\mathbf{r}(u,v)) = \mathbf{F}(S) = [3v^2, 6, 6uv] = 3(v^2\mathbf{i} + 2\mathbf{j} + 2uv\mathbf{k}).$$

Then

$$\mathbf{F}(\mathbf{r}(u,v)) \cdot \mathbf{N}(u,v) = 3[v^2, 2, 2uv] \cdot [2u, -1, 0] = 3(2uv^2 - 2) = 6(uv^2 - 1).$$

Parameters u, v belong to rectangle $R: \ 0 \le u \le 2, \ 0 \le v \le 3$. Now we can write and calculate the flux integral:

$$\int_S\int \mathbf{F} \cdot \mathbf{n}dA = \int_R\int \mathbf{F}(\mathbf{r}(u,v)) \cdot \mathbf{N}(u,v)dudv =$$

$$\int_0^3\int_0^2 6(uv^2 - 1)dudv = 6(\int_0^3 v^2dv\int_0^2 udu - \int_0^3\int_0^2 dudv) = 6(3^2 \cdot 2 - 6) = 72.$$

Example 18 Evaluation of a surface integral

Evaluate a surface integral of the vector function $\mathbf{F} = [x^2, 0, 3y^2]$ over a portion of the plane

$$S: x + y + z = 1, \ 0 \le x, y, z \le 1.$$

Solution. Writing $x = u$ and $y = v$, we have $z = 1 - u - v$ and can represent S in the form

$$\mathbf{r}(u,v) = [u, v, 1 - u - v], \quad 0 \le v \le 1, \ 0 \le u \le 1 - v.$$

We have

$$\mathbf{r}_u = [1, 0, -1], \quad \mathbf{r}_v = [0, 1, -1];$$

a normal vector is

$$\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = \mathbf{i}+\mathbf{j}+\mathbf{k} = [1,1,1].$$

The corresponding unit normal vector

$$\mathbf{n} = \frac{1}{|\mathbf{N}|}\mathbf{N} = \frac{1}{\sqrt{3}}(\mathbf{i}+\mathbf{j}+\mathbf{k}).$$

On surface S,

$$\mathbf{F}(\mathbf{r}(u,v)) = \mathbf{F}(S) = [u^2, 0, 3v^2] = u^2\mathbf{i} + 3v^2\mathbf{k}.$$

Hence

$$\mathbf{F}(\mathbf{r}(u,v)) \cdot \mathbf{N}(u,v) = [u^2, 0, 3v^2] \cdot [1,1,1] = u^2 + 3v^2.$$

Parameters u, v belong to triangle $R:\ 0 \le v \le 1,\ 0 \le u \le 1-v$. Now we can write and calculate the flux integral:

$$\int_S\int \mathbf{F}\cdot\mathbf{n}dA = \int_R\int \mathbf{F}(\mathbf{r}(u,v))\cdot\mathbf{N}(u,v)dudv = \int_R\int (u^2+3v^2)dudv =$$

$$\int_0^1\int_0^{1-v}(u^2+3v^2)dudv = \int_0^1 dv\int_0^{1-v}u^2du + 3\int_0^1 v^2dv\int_0^{1-v}du =$$

$$= (1/3)\int_0^1(1-v)^3dv + 3\int_0^1 v^2(1-v)dv = (1/3)\int_0^1 t^3dt + 3\int_0^1(v^2-v^3)dv =$$

$$(1/3)\cdot(1/4) + 3(1/3-1/4) = 1/3.$$

11.4 Problems

11.4.1 Problem *9.5.1*

Determine a normal vector and unit normal vector to the xy-plane

$$\mathbf{r}(u,v) = [u,v] = u\mathbf{i} + v\mathbf{j}$$

and parametric form of curves $u = const$ and $v = const$.

11.4.2 Problem *9.5.3*

Determine a normal vector to cone surface

$$\mathbf{r}(u,v) = u\cos v\mathbf{i} + u\sin v\mathbf{j} + cu\mathbf{k} = [u\cos v, u\sin v, cu]$$

and parametric form of curves is $u = const$ and $v = const$.

11.4.3 Problem *9.5.24*

Determine a unit normal vector to the ellipsoid $4x^2 + y^2 + 9z^2 = 36$.

12. Divergence Theorem of Gauss

Let $\mathbf{v}(x,y,z)$ be a differentiable vector function,

$$\mathbf{v}(x,y,z) = v_1(x,y,z)\mathbf{i} + v_2(x,y,z)\mathbf{j} + v_3(x,y,z)\mathbf{k},$$

then the (scalar) function

$$\operatorname{div}\mathbf{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

is called the divergence of $\mathbf{v}$.

Formulate the divergence theorem of Gauss.

Theorem 30 Let T be a closed bounded region in space whose boundary is a piecewise smooth orientable surface S. Let $\mathbf{F}(x,y,z)$ be a vector function that is continuous and has continuous first partial derivatives in some domain containing T. Then

$$\int\int_T\int \operatorname{div}\mathbf{F}dV = \int_S\int \mathbf{F}\cdot\mathbf{n}dA.$$

In components

$$\int\int_T\int \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dxdydz = \int_S\int (F_1\cos\alpha + F_2\cos\beta + F_3\cos\gamma)dA.$$

or

$$\int\int_T\int \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dxdydz = \int_S\int (F_1dydz + F_2dzdx + F_3dxdy).$$

Example 19 Evaluate

$$I = \int_S\int (x^3 dydz + x^2 ydzdx + x^2 zdxdy), \tag{12.1}$$

where S is the closed surface consisting of the cylinder $x^2+y^2=a^2$ $(0 \le z \le b)$ and the circular disks $z=0$ and $z=b$ $(x^2+y^2 \le a^2)$.

Solution.

$$F_1 = x^3, \quad F_2 = x^2 y, \quad F_3 = x^2 z.$$

Hence the divergence of $\mathbf{F} = [F_1, F_2, F_3]$ is

$$\operatorname{div}\mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 3x^2 + x^2 + x^2 = 5x^2.$$

The form of the surface suggests that we introduce polar coordinates

$$x = r\cos\theta, \quad y = r\sin\theta \quad \text{(cylindriska koordinater } r, \theta, z)$$

and

$$dxdydz = rdrd\theta dz,$$

According to Gauss's theorem, a surface integral is reduced to a triple integral if the area T is bounded by a cylindrical surface S,

$$\int_S\int (x^3 dydz + x^2 ydzdx + x^2 zdxdy) = \int\int_T\int \operatorname{div}\mathbf{F}dV = \int\int_T\int 5x^2 dxdydz =$$

$$5\int_{z=0}^{b}\int_{r=0}^{a}\int_{\theta=0}^{2\pi} r^2\cos^2\theta rdrd\theta dz =$$

$$5b\int_0^a\int_0^{2\pi} r^3\cos^2\theta drd\theta = 5b\frac{a^4}{4}\int_0^{2\pi}\cos^2\theta d\theta =$$

$$5b\frac{a^4}{8}\int_0^{2\pi}(1+2\cos\theta)d\theta = \frac{5}{4}\pi ba^4.$$

Example 20 Verification of the divergence theorem

Evaluate

$$I = \int_S\int \mathbf{F}\cdot\mathbf{n}dA, \quad \mathbf{F} = 7x\mathbf{i} - z\mathbf{k}$$

over the sphere $S: x^2+y^2+z^2 = 4$. Calculate the integral directly and using Gauss's theorem.

Solution.
$\mathbf{F}(x,y,z) = [F_1, F_2, F_3]$ is a differentiable vector function and its components are

$$\mathbf{F} = [F_1, 0, F_3], \quad F_1 = 7x, \quad F_3 = -z.$$

The divergence of $\mathbf{F}$ is

$$\operatorname{div}\mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 7 + 0 - 1 = 6.$$

Accordingly,

$$I = \int\int_{T,\text{klot}}\int \operatorname{div}\mathbf{F}dV = 6\int\int_{T,\text{klot}}\int dxdydz = 6\cdot\frac{4}{3}\pi 2^3 = 64\pi. \tag{12.2}$$

The surface integral of S can be calculated directly. Parametric representation of the sphere of radius 2

$$S: \ \mathbf{r}(u,v) = 2\cos v \cos u\mathbf{i} + 2\cos v \sin u\mathbf{j} + 2\sin v\mathbf{k},$$

$$u, v \text{ i rektangel } R: \ 0 \le u \le 2\pi, \ -\pi/2 \le v \le \pi/2.$$

Determine the partial derivatives

$$\mathbf{r}_u = [-2\sin u \cos v, 2\cos v \cos u, 0],$$

$$\mathbf{r}_v = [-2\sin v \cos u, -2\sin v \sin u, 2\cos v],$$

and the normal vector

$$\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2\sin u \cos v & 2\cos v \cos u & 0 \\ -2\sin v \cos u & -2\sin v \sin u & 2\cos v \end{vmatrix} = [4\cos^2 v \cos u, 4\cos^2 v \sin u, 4\cos v \sin v].$$

On surface S,

$$x = 2\cos v \cos u, \quad z = 2\sin v,$$

and

$$\mathbf{F}(\mathbf{r}(u,v)) = \mathbf{F}(S) = [7x, 0, -z] = [14\cos v \cos u, 0, -2\sin v].$$

Then

$$\mathbf{F}(\mathbf{r}(u,v)) \cdot \mathbf{N}(u,v) =$$

$$(14\cos v \cos u) 4\cos^2 v \cos u + (-2\sin v)(4\cos v \sin v) = 56\cos^3 v \cos^2 u - 8\cos v \sin^2 u.$$

The parameters u, v vary in the rectangle $R: 0 \le u \le 2\pi, \ -\pi/2 \le v \le \pi/2$. Now, we can write and calculate the surface integral:

$$\int_S\!\!\int \mathbf{F} \cdot \mathbf{n} dA = \int_R\!\!\int \mathbf{F}(\mathbf{r}(u,v)) \cdot \mathbf{N}(u,v) du dv =$$

$$8\int_0^{2\pi}\int_{-\pi/2}^{-\pi/2} (7\cos^3 v \cos^2 u - \cos v \sin^2 v) du dv =$$

$$8\left\{\frac{7}{2}\int_0^{2\pi}(1+\cos 2u)du \int_{-\pi/2}^{\pi/2}\cos^3 v dv - 2\pi\int_{-\pi/2}^{\pi/2}\cos v \sin^2 v dv\right\} =$$

$$56\pi\int_{-\pi/2}^{\pi/2}\cos^3 v dv - 16\pi\int_{-\pi/2}^{\pi/2}\cos v dv \sin^2 v dv =$$

$$8\pi\left\{7\int_{-\pi/2}^{\pi/2}(1-\sin^2 v) d\sin v - 2\int_{-\pi/2}^{\pi/2} dv \sin^2 v d\sin v\right\} =$$

$$8\pi\left\{7\int_{-1}^{1}(1-t^2)dt - 2\int_{-1}^{1}t^2 dt\right\} =$$

$$8\pi[7\cdot(2-2/3) - 4/3] = 8\pi\cdot 4/3\cdot 6 = 64\pi.$$

coinciding with the value (12.2).

Example 21 Physical Interpretation of divergence

The mean-value theorem for a triple integral yields

$$\int\int_T\int f(x,y,z)dV = f(x_0,y_0,z_0)V(T)$$

where (x_0,y_0,z_0) is a point on T and $V(T)$ is T's .

According to Gauss's theorem

$$\operatorname{div}\mathbf{F}(x_0,y_0,z_0) = \frac{1}{V(T)}\int\int_T\int \operatorname{div}\mathbf{F}dV = \frac{1}{V(T)}\int_{S(T)}\int \mathbf{F}\cdot\mathbf{n}dA.$$

Choose a fixed point $P:\ (x_1,y_1,z_1)$ in the T and let T shrink to P so that the maximum distance $d(T)$ between points in T and P approaches 0. As a result, one obtains a different definition of divergence

$$\operatorname{div}\mathbf{F}(x_1,y_1,z_1) = \lim_{d(T)\to 0}\frac{1}{V(T)}\int_{S(T)}\int \mathbf{F}\cdot\mathbf{n}dA.$$

This means that divergence is independent of a Cartesian coordinate system in space.

Example 22 Differential operator of the second order

$$\Delta = \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

is called the Laplace operator (Laplacian). A twice differentiable function f that satisfies the Laplace equation in domain T,

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0,$$

called *harmonic function* in T.

One can transform a double integral of Laplacian Δw to a line integral of its normal derivative $\dfrac{\partial w}{\partial n}$:

$$\int_R\int \nabla^2 w dxdy = \int_C \frac{\partial w}{\partial n}ds.$$

One can also transform a triple integral of the Δf Laplacian to a surface integral of its normal derivative for

$$\mathbf{F} = \operatorname{grad} f.$$

We have

$$\operatorname{div}\mathbf{F} = \operatorname{div}\operatorname{grad} f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \nabla^2 f,$$

and

$$\mathbf{F}\cdot\mathbf{n} = \operatorname{grad} f\cdot\mathbf{n}$$

According to Gauss divergence theorem, we get

$$\int\int_T\int \nabla^2 f dxdydz = \int_S\int \frac{\partial f}{\partial n}dA.$$

This we have shown that if $f(x,y,z)$ is a harmonic function in T ($\nabla^2 f = 0$ i T), then the integral of the normal derivative of this function is zero

$$\int_S\int \frac{\partial f}{\partial n}dA = 0.$$

12.1 Problems

12.1.1 Problem *9.7.13*

Evaluate the integral over a surface $T : |x| \leq 1, |y| \leq 3, |z| \leq 2$ då $\mathbf{F} = [x^2, 0, z^2]$.

12.1.2 Problem *9.7.15*

Evaluate the integral of $\mathbf{F} = [\cos y, \sin x, \cos z]$ where S is a closed surface consisting of the cylinder $x^2 + y^2 = 4$ ($|z| \leq 2$) and the circular disks $z = -2$ and $z = 2$ ($x^2 + y^2 \leq 4$).

12.1.3 Problem *9.8.1*

Verify the fundamental properties of solutions to the Laplace equation for $f(x,y,z) = 2z^2 - x^2 - y^2$ and S being a 'box' surface $T : 0 \leq x \leq 1, 0 \leq y \leq 2, 0 \leq z \leq 4$.

13. Stokes's Theorem

The Stokes's theorem transforms line integrals into surface integrals and generalizes Green's theorem in the plane.

Let x, y, z be a positively oriented Cartesian coordinate system and

$$\mathbf{v}(x,y,z) = v_1(x,y,z)\mathbf{i} + v_2(x,y,z)\mathbf{j} + v_3(x,y,z)\mathbf{k}$$

a differentiable vector function. The vector function

$$\operatorname{curl}\mathbf{v} = \nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix} =$$

$$\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)\mathbf{k}$$

is the curl of $\mathbf{v}$ (or rotation of vector field $\mathbf{v}$).

Let (i) S be a piecewise smooth oriented surface in space and the boundary of S a piecewise smooth simple closed curve C and (ii) $\mathbf{F}(x,y,z)$ a continuous vector function that has continuous first partial derivatives in a domain in space containing S. Then

$$\int_S\int (\operatorname{curl}\mathbf{F}) \cdot \mathbf{n}\, dA = \int_C \mathbf{F} \cdot \mathbf{r}'(s)\, ds, \tag{13.1}$$

Here $\mathbf{n}$ is a unit normal vector of S, $\mathbf{r}'(s)$ is the unit tangent vector and s the arc length of C.

Write Stokes' theorem componentwise. Remind that the parameter form of surface S is given by three equations

$$x = x(u,v), \quad y = y(u,v), \quad z = z(u,v), \qquad (u,v) \in D, \tag{13.2}$$

or by the vector function

$$\mathbf{r}(u,v) = [x(u,v), y(u,v), z(u,v)] = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k} \quad [(u,v) \in D], \tag{13.3}$$

here $\mathbf{r}(u,v)$ is the position vector of the point $P = P(u,v) = (x(u,v), y(u,v), z(u,v)) \in S$ and variables u, v are parameters. Domain D is located in the uv-plane and is called the parameter domain.

We can write Stoke's theorem in components

$$\int_R\int \left[\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)N_1 + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)N_2 + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)N_3\right] dudv =$$

$$= \int_{\tilde{C}} (F_1 dx + F_2 dy + F_3 dz),$$

where R is the parameter domain in the uv-plane bounded by spatial curve $\tilde{C}$ corresponding to S with parameter vector function $\mathbf{r}(u,v)$ and normal vector $\mathbf{N}(u,v) = [N_1, N_2, N_3] = \mathbf{r}_u \times \mathbf{r}_v$.

Example 23 Verification of Stokes's Theorem

Verify Stokes' theorem for the vector functions

$$\mathbf{F} = [y, z, x] = y\mathbf{i} + z\mathbf{j} + x\mathbf{k}$$

and the surface (paraboloid) S given by

$$z = f(x,y) = 1 - (x^2 + y^2), \quad z \geq 0$$

Solution. The curve C is a circle $\mathbf{r}(s) = [\cos s, \sin s, 0]$. Its unit tangent vector is $\mathbf{r}'(s) = [-\sin s, \cos s, 0]$. Hence

$$\int_C \mathbf{F} \cdot \mathbf{r}'(s) ds = \int_0^{2\pi} [\sin s(-\sin s) + 0 + 0] ds = -\pi.$$

According to Stokes's theorem, we get

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} =$$

$$\left(\frac{\partial x}{\partial y} - \frac{\partial z}{\partial z}\right)\mathbf{i} + \left(\frac{\partial y}{\partial z} - \frac{\partial x}{\partial x}\right)\mathbf{j} + \left(\frac{\partial z}{\partial x} - \frac{\partial y}{\partial y}\right)\mathbf{k} =$$
$$-\mathbf{i} - \mathbf{j} - \mathbf{k} = [-1, -1, -1].$$

A normal vector of S is

$$\mathbf{N} = \operatorname{grad}(z - f(x,y)) = [2x, 2y, 1].$$

Furthermore, we calculate the scalar product and get

$$\operatorname{curl} \mathbf{F} \cdot \mathbf{N} = [-1, -1, -1] \cdot [2x, 2y, 1] = -2x - 2y - 1.$$

Using polar coordinates defined by $x = r\cos\theta$, $y = r\sin\theta$: we obtain

$$\int_S\int (\operatorname{curl} \mathbf{F}) \cdot \mathbf{n} dA = \int_R\int (-2x - 2y - 1) dxdy = \int_{\tilde{R}}\int (-2r\cos\theta - 2r\sin\theta - 1) r dr d\theta,$$

where $\tilde{R}$ is the circle $r \leq 1$, $0 \leq \theta \leq 2\pi$. Next we have

$$\int_{\tilde{R}}\int (-2r\cos\theta - 2r\sin\theta - 1) r dr d\theta =$$

$$-2\int_0^{2\pi} \cos\theta d\theta \int_0^1 r dr - 2\int_0^{2\pi} \sin\theta d\theta \int_0^1 r dr - \int_0^{2\pi} d\theta \int_0^1 r dr = 0 + 0 - \pi = -\pi.$$

Example 24 Evaluation of a line integral by Stokes's theorem

Evaluate

$$\int_C \mathbf{F}\cdot\mathbf{r}'(s)ds,$$

where

$$\mathbf{F} = [y, xz^3, -zy^3] = y\mathbf{i} + xz^3\mathbf{j} - zy^3\mathbf{k},$$

and C is the circle

$$x^2+y^2=4, \quad z=-3.$$

Calculate the integral directly and with the help of Stokes' theorem.

Solution. As a surface S bounded by C we can take the plane circular disk $S: x^2+y^2 \le 4$ in the plane $z=-3$. Then a normal vector to S $\mathbf{n}=\mathbf{k}=[0,0,1]$, and Stokes' theorem gives

$$\operatorname{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & xz^3 & -zy^3 \end{vmatrix} =$$

$$\left(\frac{\partial(-zy^3)}{\partial y} - \frac{\partial(xz^3)}{\partial z}\right)\mathbf{i} + \left(\frac{\partial y}{\partial z} - \frac{\partial(-zy^3)}{\partial x}\right)\mathbf{j} + \left(\frac{\partial(xz^3)}{\partial x} - \frac{\partial y}{\partial y}\right)\mathbf{k}$$

$$-3z(y^2+xz)\mathbf{i} + (z^3-1)\mathbf{k} = [-3z(y^2+xz), 0, z^3-1].$$

$$\operatorname{curl}\mathbf{F}\cdot\mathbf{N} = \operatorname{curl}\mathbf{F}\cdot\mathbf{k} = z^3-1$$

and

$$\operatorname{curl}\mathbf{F}\cdot\mathbf{N}|_{z=-3} = -3^3-1 = -28.$$

Then

$$\int_S\int(\operatorname{curl}\mathbf{F})\cdot\mathbf{n}dA = \int_{x^2+y^2\le 4\; z=-3}\int(-28)dxdy = -28\pi 2^2 = -112\pi.$$

13.1 Problems

13.1.1 Problem *9.9.1*

Evaluate the surface integral

$$\int_S\int(\operatorname{curl}\mathbf{F})\cdot\mathbf{n}dA,$$

where $\mathbf{F}=[z^2, 5x, 0]$ and S is a square

$$0\le x\le 1,\ 0\le y\le 1,\ z=1.$$

13.1.2 Problem *9.9.3*

Evaluate the surface integral

$$\int_S\int(\operatorname{curl}\mathbf{F})\cdot\mathbf{n}dA,$$

where $\mathbf{F}=[e^z, e^z\sin y, e^z\cos y]$ and S is a cylindrical paraboloid.

$$z=y^2,\ 0\le x\le 4\ 0\le y\le 2.$$

14. BVPs for the Laplace equation

Assume we have a rectangular plate of length a and width b of constant thermal conductivity. Assume further that the flat domain is isolated and that the sides are at constant or non-constant temperature described by four given functions $f_1(y)$, $f_2(y)$, $0 < y < b$, and $f_3(x)$, $f_4(x)$, $0 < x < a$. Let $u = u(x,y,t)$ denote the temperature of the point (x,y), $0 < x < a$, $0 < y < b$, and the time moment $t \geq 0$. After a certain sufficiently long time interval, the plate temperature tends to be stationary i.e. independent of time. Then $u = u(x,y)$ satisfies the BVP for the two-dimensional Laplace equation in the rectangle $\Pi = \{x,y) : 0 < x < a, 0 < y < b\}$ $(\bar{\Pi} = \{x,y) : 0 \leq x \leq a, 0 \leq y \leq b\})$

$$\begin{cases} \Delta u = 0, \quad u = u(x,y), \quad 0 < x < a, \quad 0 < y < b, \\ u \in C^2(\Pi) \cap C(\bar{\Pi}), \\ u(0,y) = f_1(y), \qquad u(a,y) = f_2(y),\ 0 \leq y \leq b, \\ u(x,0) = f_3(x), \qquad u(x,b) = f_4(x),\ 0 \leq x \leq a, \\ f_{3,4}(x) \in C([0,a]), \quad f_{1,2}(y) \in C([0,b]). \end{cases} \tag{14.1}$$

Note that the corresponding homogeneous BVP

$$\begin{cases} \Delta u = 0, \quad u = u(x,y), \quad 0 < x < a, \quad 0 < y < b, \\ u \in C^2(\Pi) \cap C(\bar{\Pi}), \\ u(0,y) = 0, \qquad u(a,y) = 0, \\ u(x,0) = 0, \qquad u(x,b) = 0 \end{cases}$$

has only a trivial solution.

14.1 Separation of variables in a rectangle

Determine a solution $u \in C^2(\Pi) \cap C(\bar{\Pi})$ to the Laplace equation $\Delta u = 0$ which has the form

$$u(x,y) = X(x)Y(y). \tag{14.2}$$

Insert (14.2) in $\Delta u = 0$:

$$X''Y + Y''X = 0, \tag{14.3}$$

$$\frac{X''}{X} + \frac{Y''}{Y} = 0. \tag{14.4}$$

Both fractions in (14.4) must be equal to a constant. We consider two cases I and II, which correspond to a negative and a positive constant denoted by $-\lambda^2 < 0$ and $\mu^2 > 0$.

Solve the first two differential equations for $X(x)$ and $Y(y)$ arising from the (14.3) and (14.4) with a negative constant $-\lambda^2 < 0$:

$$\text{case I} \quad \frac{X''}{X} = -\lambda^2 \Rightarrow \quad X'' + \lambda^2 X = 0 \quad \Rightarrow \quad X(x) = B_1 \cos\lambda x + B_2 \sin\lambda x$$
$$\frac{Y''}{Y} = \lambda^2 \Rightarrow \quad Y'' - \lambda^2 Y = 0 \quad \Rightarrow \quad Y(y) = C_1 \cosh\lambda y + C_2 \sinh\lambda y$$

and with a positive constant $\mu^2 > 0$:

$$\text{case II} \quad \frac{X''}{X} = \mu^2 \Rightarrow \quad X'' - \lambda^2 X = 0 \quad \Rightarrow \quad X(x) = D_1 \cosh\lambda x + D_2 \sinh\lambda x, \tag{14.5}$$
$$\frac{Y''}{Y} = -\mu^2 \Rightarrow \quad Y'' + \lambda^2 Y = 0 \quad \Rightarrow \quad Y(y) = E_1 \cos\lambda y + E_2 \sin\lambda y. \tag{14.6}$$

Now use (14.5), (14.6) and its equivalent in cases I and determine solutions to the Laplace equation that satisfy homogeneous boundary conditions (14.16) on all sides of the rectangle Π except one. We have four such cases:

Homogeneous boundary conditions (14.16) on all sides of Π except $y = b$

$$\begin{aligned} 1 \qquad X(0) &= X(a) = 0 \quad \Rightarrow \quad B_1 = 0, B_2 = 1; \lambda_n = \frac{\pi n}{a} = \alpha n; C_1 = 0 \\ Y(0) &= 0 \qquad \mathbf{u} = \mathbf{u_n}(\mathbf{x},\mathbf{y}) = \sin\alpha n x \sinh\alpha n y \\ u_n(0,y) &= u_n(a,y) = 0; \quad u_n(x,0) = 0. \end{aligned} \tag{14.7}$$

Homogeneous boundary conditions (14.16) on all sides of Π except $y = 0$

$$\begin{aligned} 2 \qquad X(0) = X(a) = 0 \quad &\Rightarrow \quad B_1 = 0, B_2 = 1; \lambda_n = \frac{\pi n}{a} = \alpha n; \\ &C_1 = \sinh(\alpha n b); C_2 = -\cosh(\alpha n b) \\ Y(b) = 0 \qquad &\mathbf{u} = \mathbf{u_n}(\mathbf{x},\mathbf{y}) = \sin\alpha n x \sinh\alpha n(b - y), \\ &u_n(0,y) = u_n(a,y) = 0; \quad u_n(x,b) = 0. \end{aligned} \tag{14.8}$$

Homogeneous boundary conditions (14.16) on all sides of Π except $x = a$

$$\begin{aligned} 3 \qquad Y(0) &= Y(b) = 0 \quad \Rightarrow \quad E_1 = 0; E_2 = 1; \mu_n = \frac{\pi m}{b} = \beta m; D_1 = 0 \\ X(0) &= 0 \qquad \mathbf{u} = \mathbf{u_m}(\mathbf{x},\mathbf{y}) = \sinh\beta m x \sin\beta m y, \\ u_m(x,0) &= u_m(x,b) = 0; \quad u_m(0,y) = 0. \end{aligned} \tag{14.9}$$

Homogeneous boundary conditions (14.16) on all sides of Π except $x=0$

$$
\begin{aligned}
4 \qquad Y(0) &= Y(b) = 0 \quad \Rightarrow \quad E_1 = 0; E_2 = 1; \mu_n = \frac{\pi m}{b} = \beta m; D_1 = 0 \\
X(a) &= 0 \qquad \mathbf{u} = \mathbf{u_m}(\mathbf{x},\mathbf{y}) = \sinh \beta m(a-x) \sin \beta m y, \qquad (14.10) \\
u_m(x,0) &= u_m(x,b) = 0; \quad u_m(a,y) = 0.
\end{aligned}
$$

u given by (14.7)–(14.10) are solutions to the Laplace equation in a rectangle Π which have the form (14.2). One can use (14.7)–(14.10) and determine the solution to (14.1) as a Fourier series using the Fourier method.

14.2 Dirichlet BVP in a rectangle with a given boundary function

Consider an example of the BVP (14.1)

$$
\begin{cases}
\Delta u = 0, \quad u = u(x,y), \quad 0 < x < a, \quad 0 < y < b, \\
u(0,y) = 0, \qquad u(a,y) = 0, \quad 0 \le y \le b, \\
u(x,0) = 0, \qquad u(x,b) = H(x), \quad 0 \le x \le a,
\end{cases} \qquad (14.11)
$$

where $u \in C^2(\Pi) \cap C(\bar{\Pi})$, with a boundary function

$$H(x) \in C^1[0,a] \cap L = supp\, H(x) \subset (0,a)$$

dvs $H(x_S \pm p) = 0$, $H'(x_S \pm p) = 0$ and $H(x) = 0$, $x \notin L = supp\, H(x) = (x_S - p, x_S + p)$, $0 < x_S < a$; for example

$$
H(x) = \begin{cases}
Q[p^2 - (x-x_S)^2]^2 e^{-r(x-x_S)^2}, \ |x - x_S| \le p, \\
0, \quad |x - x_S| \ge p,
\end{cases} \qquad (14.12)
$$

where p, Q and r are given positive numbers, and $supp\, H(x) = L = (x_S - p, x_S + p) \subset (0,a)$.

14.3 Problem

Solve the BVP

$$
\begin{cases}
\Delta u = 0, \quad u = u(x,y), \quad 0 < x < a, \quad 0 < y < b, \\
u(0,y) = 0, \qquad u(a,y) = 0, \quad 0 \le y \le b, \\
u(x,0) = H(x), \qquad u(x,b) = 0, \quad 0 \le x \le a,
\end{cases} \qquad (14.13)
$$

with a boundary function $H(x) \in C^1[0,a] \cap L = supp\, H(x) \subset (0,a)$ ie $H(x_S \pm p) = 0$, $H'(x_S \pm p) = 0$ and $H(x) = 0$, $x \notin L = supp\, H(x) = (x_S - p, x_S + p)$, $0 < x_S < a$. Use the Fourier method and determine solution as a Fourier series.

Solve the BVP

$$
\begin{cases}
\Delta u = 0, \quad u = u(x,y), \quad 0 < x < a, \quad 0 < y < b, \\
u(0,y) = H(y), \qquad u(a,y) = 0, \quad 0 \le y \le b, \\
u(x,0) = 0, \qquad u(x,b) = 0, \quad 0 \le x \le a,
\end{cases} \qquad (14.14)
$$

with a boundary function $H(y) \in C^1[0,b] \cap L = supp\, H(y) \subset (0,b)$ ie $H(y_S \pm p) = 0$, $H'(y_S \pm p) = 0$ and $H(x) = 0$, $y \notin L = supp\, H(y) = (y_S - p, y_S + p)$, $0 < y_S < b$. Use the Fourier method and determine solution as a Fourier series.

14.4 BVPs for the Laplace and Poisson equations

Consider a BVP for the two-dimensional Laplace equation in the rectangle with vertexes $(0,0)$, $(a,0)$, (a,b) and $(0,b)$:

$$\begin{cases} \Delta u = 0, \quad u = u(x,y), \quad 0 < x < a, \quad 0 < y < b, \\ u(0,y) = f_1(y), \qquad u(a,y) = f_2(y), \\ u(x,0) = f_3(x), \qquad u(x,b) = f_4(x). \end{cases} \tag{14.15}$$

The two-dimensional Poisson equation is

$$\Delta u = \rho(x,y), \quad u = u(x,y), \quad 0 < x < a, \quad 0 < y < b. \tag{14.16}$$

Note that e.g. the temperature is defined by a continuous function in the closed rectangle. Thus boundary functions f_i in (14.15) must coinside in the corner points $(0,0)$, $(a,0)$, (a,b) and $(0,b)$:

$$\begin{aligned} u(0,0) = f_1(0) = f_3(0), \qquad u(a,0) = f_3(a) = f_2(0), \\ u(0,b) = f_1(b) = f_4(0), \qquad u(a,b) = f_4(a) = f_2(b). \end{aligned} \tag{14.17}$$

Consider an example: the functions

$$f_1(y) = y(1-y), \quad f_2(y) = 2y(1-y), \quad f_3(x) = x(1-x), \quad f_4(x) = 3x(1-x) \tag{14.18}$$

satisfy (14.17) on the sides of a square $0 < x < 1$, $0 < y < 1$:

$$\begin{aligned} f_1(0) = f_3(0) = 0, \qquad f_3(1) = f_2(0) = 0, \\ f_1(1) = f_4(0) = 0, \qquad f_4(1) = f_2(1) = 0. \end{aligned}$$

The numerical solution to the corresponding BVP (14.15) for the Laplace equation in the unit square is illustrated by Fig. 15.3.

Fig. 15.4 shows the results of the numerical solution to the BVP for the Poisson equation in the unit square

$$\begin{cases} \Delta u = \rho(x,y), \quad u = u(x,y), \quad 0 < x < 1, \quad 0 < y < 1, \\ u(0,y) = 0, \qquad u(a,y) = 0, \\ u(x,0) = 0, \qquad u(x,b) = 0. \end{cases} \tag{14.19}$$

The right-hand side $\rho(x,y)$ is shown in Fig. 15.5.

15. Numerical solution of BVPs for the Laplace and Poisson equations

15.1 Approximation

To approximate the function $u(x,y)$, $x \in [0,a]$, $y \in [0,b]$, and solve the BVP (14.1) numerically by replacing it with a differential approximation divide intervals $[0,a]$ and $[0,b]$ in (smaller) N_x and N_y intervals, respectively, and specify functions' $(N_x - 1)$ and $(N_y - 1)$ x- and y-values at the nodes $x_i = ih_x$ and $y_j = jh_y$ equally spaced with the distances $h_x = \frac{a}{n}$ and $h_y = \frac{b}{N_y}$:

$$\begin{aligned} x_i &= ih_x, \quad i = 0,1\dots,N_x, \quad x_0 = 0 < x_1 < x_2 < \cdots < x_{N_x-1} < x_{N_x} = a, \\ y_j &= jh_y, \quad j = 0,1\dots,N_y, \quad y_0 = 0 < y_1 < y_2 < \cdots < y_{N_y-1} < y_{N_y} = b,. \end{aligned} \tag{15.1}$$

We approximate the Laplace and Poisson equations in (14.15) and (14.16) with the finite differences at the points (x_i, y_j), where

$$\begin{gathered} x_i = x_0 + ih_x = ih_x, \quad y_j = y_0 + jh_y = jh_y, \\ i = 1,2,\dots,N_x - 1, \qquad j = 1,2,\dots,N_y - 1, \end{gathered} \tag{15.2}$$

by the finite-difference expressions

$$\begin{gathered} \frac{u_{i-1,j} - 2u_{i,j} + u_{i-1,j}}{h_x^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h_y^2} = \rho_{i,j}, \\ i = 1,2,\dots,N_x - 1, \qquad j = 1,2,\dots,N_y - 1, \end{gathered} \tag{15.3}$$

where $\rho_{i,j} = \rho(x_i, y_j)$ (Poisson equation) or $\rho_{i,j} = 0$ (Laplace equation).

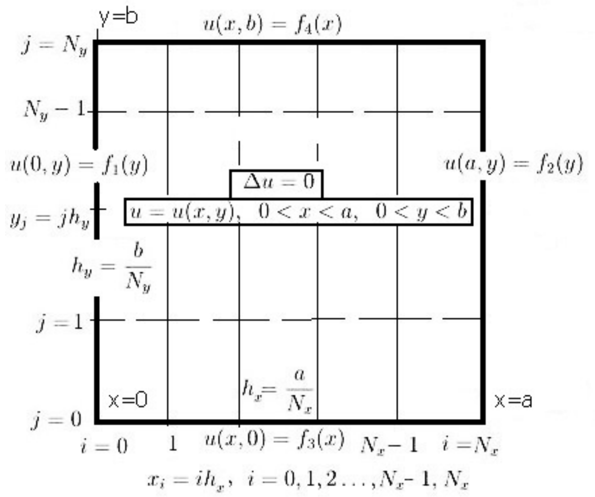

Figure 15.1: Numerical solution of BVPs for the Laplace and Poisson equations in a rectangle.

Now suppose $h_x=h_y=h$. Rewriting(15.3)

$$u_{i,j}=\frac{1}{4}[u_{i+1,j}+u_{i-1,j}+u_{i,j+1}+u_{i,j-1}]-h^2\rho_{i,j}, \tag{15.4}$$

we get $(N_x-1)(N_y-1)$ linear equations with unknowns $u_{i,j}$. To obtain the corresponding linear equation system, write $u_{i,j}$ as a vector

$$u_{i,j}=u_k, \quad k=1,2,\ldots,N=(N_x-1)(N_y-1). \tag{15.5}$$

We count unknowns under the rule

$$k=j+(N_y-1)(i-1), \quad k=1,2,\ldots,N=(N_x-1)(N_y-1), \tag{15.6}$$

so that

$$\begin{array}{rcl} k &=& 1 \quad 2 \quad \ldots \quad N_y-1 \\ (i,j) &=& (1,1)(1,2)\ldots(1,N_y-1) \\ k &=& N_y\ N_y+1\ \ldots\ 2N_y-2 \\ (i,j) &=& (2,1)(2,2)\ldots(2,N_y-1) \\ (i,j) &=& \ldots \end{array}$$

(15.4) rewriting

$$u_{k+1}-4u_k+u_{k-1}+u_{(N_y-1)+k}+u_{(N_y-1)-k}=h^2\rho_k, \quad k=1,2,\ldots,N=(N_x-1)(N_y-1). \tag{15.7}$$

Use (15.7) and write each equation at a point (x_i,y_j), $i=1,2,\ldots,N_x-1$; for example, at $i=1$,

$$\begin{array}{rcll} j &=& 1 & -4u_1+u_2+u_{N_y}=-[f_1(y_1)+f_3(x_1)] \\ j &=& 2 & u_1-4u_2+u_3+u_{N_y+1}=-f_3(x_2) \\ j &=& 3 & u_2-4u_3+u_4+u_{N_y+2}=-f_3(x_3) \\ . & . & \ldots & \\ j &=& N_y-1 & u_{N_y-1}-4u_{N_y}+u_{2N_y}=-[f_2(y_1)+f_3(x_{N_x})] \end{array}$$

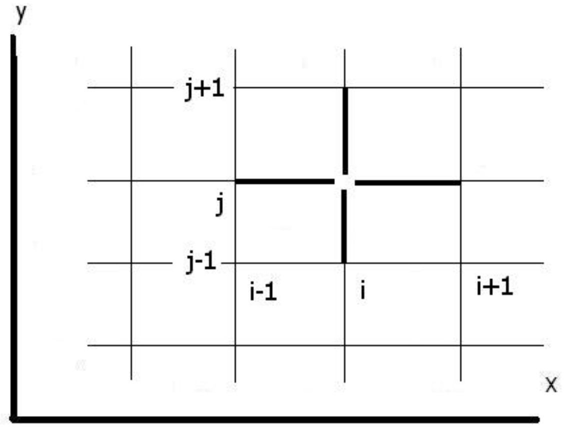

Figure 15.2: Difference approximation of the Laplace operator.

Finally, the system of equations (15.7) turns to a system of linear equations

$$\mathrm{A}\mathbf{u} = \mathbf{f} \tag{15.8}$$

with a symmetrical tridiagonal block matrix

$$\mathrm{A} = \begin{bmatrix} \mathbf{R} & \mathbf{I} & 0 & \dots & 0 \\ \mathbf{I} & \mathbf{R} & \mathbf{I} & \dots & 0 \\ 0 & \mathbf{I} & \mathbf{R} & \dots & 0 \\ . & . & . & \dots & . \\ . & . & . & \dots & . \\ . & . & . & \dots & . \\ 0 & 0 & 0 & \dots & \mathbf{I} \\ 0 & 0 & 0 & \dots & \mathbf{R} \end{bmatrix}, \tag{15.9}$$

$\mathbf{R}$ is an $N \times N$ tridiagonal matrix,

$$\mathbf{R} = \begin{bmatrix} -4 & 1 & 0 & \dots & 0 \\ 1 & -4 & 1 & \dots & 0 \\ 0 & 1 & -4 & \dots & 0 \\ . & . & . & \dots & . \\ . & . & . & \dots & . \\ . & . & . & \dots & . \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & -4 \end{bmatrix}, \tag{15.10}$$

$\mathbf{I}$ is a unit matrix of size $N \times N$,

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ . & . & . & \dots & . \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix},$$

u is a column vector of size N,

$$\mathbf{u} = \{u_k\}^T, \quad k = 1, 2, \ldots, N, \tag{15.11}$$

and **f** is a sum of two column vectors of size N,

$$\mathbf{f} = \mathbf{f}_1 + \mathbf{f}_2, \tag{15.12}$$

where

$$\mathbf{f}_1 = [f_3(\mathbf{x}) \quad 0 \quad 0\ldots0 \quad 0 \quad f_4(\mathbf{x})]^T,$$

$$\mathbf{f}_2 = [f_1(y_1) \quad 0\ldots0 \quad f_2(y_1) \quad f_1(y_2) \quad 0\ldots0 \quad f_2(y_2) \quad f_1(y_3) \quad 0\ldots0 \quad f_2(y_{N_y})]^T, \tag{15.13}$$

$$\mathbf{x} = \{x_i = ih\}, \quad i = 1, 2, \ldots, N_x.$$

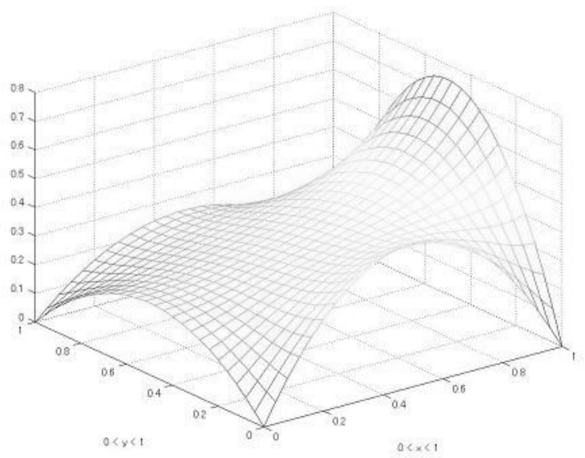

Figure 15.3: Example of numerical solution to BVP (14.15) for Laplace equation in the unit square $0 < x < 1$, $0 < y < 1$. Boundary functions are defined in (14.17).

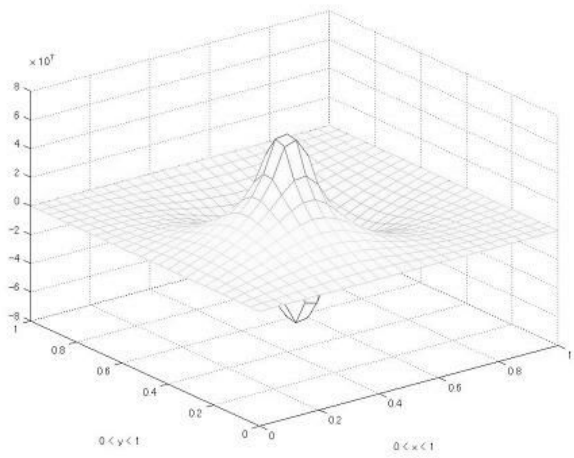

Figure 15.4: Example of numerical solution to BVP (14.19) for Poisson equation $\Delta u = \rho(x,y)$ in the unit square. Right-hand side $\rho(x,y)$ is shown in Fig. 15.5.

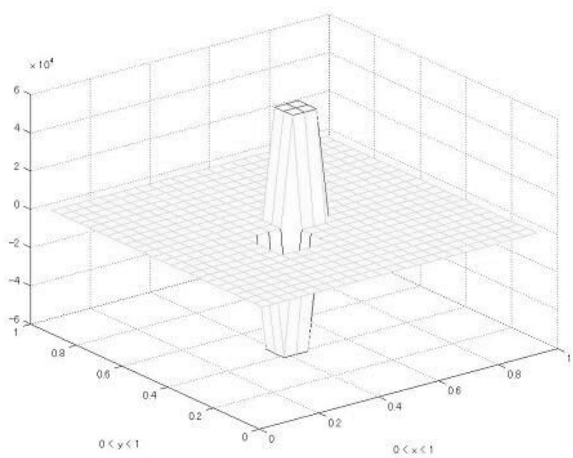

Figure 15.5: Right-hand side $\rho(x,y)$ of Poisson equation $\Delta u = \rho(x,y)$ in the unit square.

16. Introduction to the finite element method

16.1 Piecewise linear elements

An n-dimensional vector

$$\mathbf{a} = [a_1, a_2, \dots, a_n]$$

is defined as an element of an n-dimensional space R^n and is an ordered set of n components $a_1, a_2, \dots, a_n$. The n vectors

$$\mathbf{i_1} = [1,0,0,\dots,0], \quad \mathbf{i_2} = [0,1,0,\dots,0], \quad \dots, \quad \mathbf{i_n} = [0,\dots,0,1]. \tag{16.1}$$

form an (orthonormal) basis in R^n. Each vector a = $\mathbf{a} = [a_1, a_2, \dots, a_n] \in \mathrm{R}^n$ can be written as a linear combination of the basis vectors,

$$\mathbf{a} = a_1\mathbf{i_1} + a_2\mathbf{i_2} + \dots, + a_n\mathbf{i_n}. \tag{16.2}$$

To introduce the piecewise linear finite elements, divide, as in (15.1), interval $[0,1]$ in M (smaller) intervals $K_j = [x_{j-1}, x_j]$, $j = 1,2,\dots,M$ ($M \geq 2$), with the points

$$x_0 = 0 < x_1 < x_2 < \cdots < x_{M-1} < x_M = 1.$$

(in general, nonuniformly distributed with different distances between them $h_j = x_j - x_{j-1}$, $j = 1,2,\dots,M$). The corresponding $(M+1)$-dimensional vector

$$\mathbf{X_M} = [x_0, x_1, x_2, \dots, x_{M-1}, x_M] \tag{16.3}$$

is called *partition* of the base interval $[0,1]$.

Note that for the points x_j uniformly distributed with the distance $h = \dfrac{1}{n}$,

$$\begin{aligned} x_j &= jh, \quad j = 0,1\dots,M, \\ x_0 &= 0 < x_1 = h < x_2 = 2h < \cdots < x_{M-1} = (M-1)h < x_M = Mh = 1. \end{aligned} \tag{16.4}$$

The corresponding partition

$$\mathbf{X_M} = [0, h, 2h, \dots, (M-1)h, 1] = h[0,1,2,\dots,M-1,M]. \qquad (16.5)$$

The piecewise linear elements are defined as

$$\Phi_j(x) = \begin{cases} 0 & x_0 \le x \le x_{j-1}, \\ \frac{x-x_{j-1}}{h_j} & x_{j-1} \le x \le x_j, \\ \frac{x_{j+1}-x}{h_{j+1}} & x_j \le x \le x_{j+1}, \\ 0 & x_{j+1} \le x \le x_M, \end{cases} \quad j = 1,2\dots M-1,$$
$$h_j = x_j - x_{j-1}, \quad j = 1,2\dots,M. \qquad (16.6)$$

Each $\Phi_j(x)$ is a piecewise linear 'rectangular' function such that

$$\Phi_i(x_j) = \begin{cases} 1 & i = j, \\ 0 & i \ne j; \end{cases}, \quad i,j = 1,2\dots M-1, \qquad (16.7)$$

it does not equal 0 in each subinterval $[x_{j-1}, x_{j+1}] = K_j \cup K_{j+1}$, $j = 1,2,\dots,M-1$ (see Fig. 16.1).

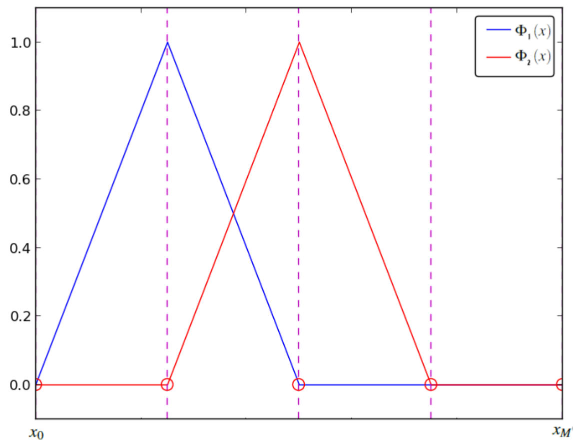

Figure 16.1: The piecewise linear elements.

Assume that $\mathbf{X_M} = [x_0, x_1, x_2, \dots, x_{M-1}, x_M]$ is a given partition of $[0,1]$ into M subinterval $K_j = [x_{j-1}, x_j]$, $j = 1,2,\dots,M$ $(M \ge 2)$. Define the $(M-1)$-dimensional space $S_h = S_h(\mathbf{X_M})$ of piecewise linear functions

$$S_h = \{v \in S_h : \ v \text{ a linear in each subinterval } K_j,\ v(0) = v(1) = 0,\ h = \max h_j\}. \qquad (16.8)$$

Theorem 31 The set $\{\Phi_j(x)\}$ of piecewise linear elements is a basis in space S_h; i.e., any piecewise linear function can be written as a linear combination of $\Phi_j(x)$.

Proof. A piecewise linear function $F = F(M;x)$ defined on the interval $[0,1]$ is a linear function on each subinterval $K_j = [x_{j-1}, x_j]$, $j = 1,2,\dots,M$. This function vanishes on the endpoints of interval $[0,1]$ so, that $F(M,0) = F(M,1) = 0$ and has $M-1$ vertices and its derivative is undefined in these points.

Thus the function $F = F(M;x)$ is composed of M piecewise linear functions $F_j(x)$,

$$F(M;x) = \begin{cases} \dots & \dots, \\ F_j(x), & x \in K_j, \\ \dots & \dots, \end{cases}, \quad j = 1,2\dots M, \qquad (16.9)$$

Function $F=F(M;x)\in S_h$ has values T_j in nodes x_j, $j=1,2,\dots,M-1$ (i.e., the function goes through the points (x_j,T_j), $j=0,1,2,\dots,M$) and for endpoints of the interval is defined as $F(M,0):=T_0=0$, and $F(M,1):=T_M=0$, respectively. Thus any subfunction $F_j(x)$ goes through the points (x_{j-1},T_{j-1}), (x_j,T_j), and uniquely determined on each subinterval $K_j=[x_{j-1},x_j]$ (as a linear function) by

$$F_j(x_{j-1})=T_{j-1},\quad F_j(x_j)=T_j, \tag{16.10}$$

We obtain that any piecewise linear function $F=F(M;x)\in S_h$ which has values T_j in the nodes x_j is uniquely determined on the interval $[0,1]$ under the conditions

$$F_j(x_j)=T_j,\quad j=0,1,2,\dots,M,\quad T_0=T_M=0. \tag{16.11}$$

Now let us show that any given piecewise linear function $F=F(M;x)\in S_h$ which has values T_j in the nodes x_j, $j=0,1,2,\dots,M$, with $T_0=T_M=0$ is a linear combination of piecewise linear base elements $\Phi_j(x)$. A linear combination of $\Phi_j(x)$ is

$$\tilde{F}(x)=\sum_{i=1}^{M-1}T_i\Phi_i(x). \tag{16.12}$$

$\tilde{F}(x)$ is a piecewise linear function (as a sum of piecewise linear functions) and

$$\tilde{F}(x_j)=\sum_{i=1}^{M-1}T_i\Phi_i(x_j)=T_j \tag{16.13}$$

$$\tilde{F}(x_0)=\sum_{i=1}^{M-1}T_i\Phi_i(x_0)=0,\quad \tilde{F}(x_M)=\sum_{i=1}^{M-1}T_i\Phi_i(x_0)=0, \tag{16.14}$$

according to (16.7), so

$$\tilde{F}(x)=\sum_{i=1}^{M-1}T_i\Phi_i(x)=F(M;x)\in S_h. \tag{16.15}$$

Consider a $(M-1)$-dimensional space $S_h=(\mathbf{X_m})$ of piecewise linear functions. The minimal value of parameter $M=2$ gives us two subintervals $K_1=[x_0,x_1]$ and $K_2=[x_1,x_2]$; Then the corresponding partition

$$\mathbf{X_2}=[x_0,x_1,x_2]=[0,x_1,1] \tag{16.16}$$

is a 3-dimensional vector. For this partition, we can define only piecewise linear 'triangular' elements $\Phi_1(x)$ by formula (16.6) for $j=1$

$$\Phi_1(x)=\begin{cases}\frac{x-x_0}{h_1}=\frac{x}{h_1}=\frac{x}{x_1} & 0=x_0\le x\le x_1,\\ \frac{x_2-x}{h_2}=\frac{1-x}{h_2}=\frac{1-x}{1-x_1} & x_1\le x\le x_2=1,\end{cases}$$
$$h_1=x_1-x_0=x_1,\quad h_2=x_2-x_1=1-x_1, \tag{16.17}$$

which satisfies (according to (16.7))

$$\Phi_1(x_1)=1,\quad \Phi_1(x_0)=\Phi_1(0)=0,\quad \Phi_1(x_2)=\Phi_1(1)=0, \tag{16.18}$$

and not equal to 0 on the whole interval $[x_0; x_2] = K_1 \cup K_2 = [0,1]$. In this case $M = 2$ and the basic element $\Phi_1(x)$ (16.17) is an element of one-dimensional space $S_h = (\mathbf{x_2})$ which consists of one piecewise linear 'triangular' functions $v(x) := C\Phi_1(x)$ with an arbitrary C:

$$S_h = S_h(\mathbf{X_2}) = \{C\Phi_1(x) \quad \forall C \in \mathrm{R}\},$$

$$v(x) \in S_h(\mathbf{X_2}): \ v(x_1) = C, \quad v(x_0) = v(0) = 0, \quad v(x_2) = v(1) = 0. \tag{16.19}$$

In the same way one can show that in the case $M > 2$, the $(M-1)$-dimensional space $S_h = S_h(\mathbf{X_M})$ consisting of piecewise linear functions which take values T_j in nodes x_j, $j = 1,2,\dots,M-1$ and vanishes in nodes $x_0 = 0$ and $x_M = 1$ (and can be written as (16.12)),

$$S_h = S_h(\mathbf{X_M}) = \left\{ \sum_{i=1}^{M-1} T_i \Phi_1(x), \quad \forall \mathbf{T_M} = [T_1, T_2, \dots, T_{M-1}] \right\}. \tag{16.20}$$

We can determine piecewise linear base elements $\Phi_j(x) \in S_h(\mathbf{X_M})$ with the base vectors (16.1) and a piecewise linear function $F = F(M;x) \in S_h(\mathbf{X_M})$ which takes values T_j, $T_0 = T_M = 0$, in nodes x_j, $j = 0,1,2,\dots,M-1,M$. The $(M-1)$-dimensional vector of the values is

$$\mathbf{T_M} = [T_1, T_2, \dots, T_{M-1}] \tag{16.21}$$

The set $C_0^1(\bar{I}_0)$ denotes a set of continuously differentiable in the closed interval $\bar{I}_0 = [0,1]$ functions $f(x)$ which satisfy the following boundary conditions

$$f(x) \in C_0^1(\bar{I}_0): \quad f(0) = 0, \ f(1) = 0. \tag{16.22}$$

A projection $P_M(f)$ of a function $f(x) \in C_0^1(\bar{I}_0)$ in the $(M-1)$-dimensional space $S_h = S_h(\mathbf{X_M})$ of piecewise linear functions with respect to a given partition (16.3) $\mathbf{X_M} = [x_0, x_1, \dots, x_M]$ $(M \geq 2)$ is defined as (16.12)

$$P_M(f) = \sum_{i=1}^{M-1} f(x_i)\Phi_i(x). \tag{16.23}$$

We can determine the projection $P_M(f)$ as a $(M-1)$-dimensional vector

$$\mathbf{P_M} = [f_1, f_2, \dots, f_{M-1}], \quad f_i = f(x_i). \tag{16.24}$$

16.2 Numerical solution of BVPs using the finite element method

Consider a BVP for a linear differential equation of the second order

$$\begin{cases} Ay = -(ay')' + q(x)y = f(x), & x \in I_0 = (0,1), \\ y(0) = 0, \quad y(1) = 0, \end{cases} \tag{16.25}$$

where $a(x)$, $q(x)$ and $f(x)$ are smooth functions satisfying the following conditions

$$a(x) \geq a_0 > 0, \quad q(x) \geq 0.$$

16.2.1 Variational formulation

The Variation formulation of BVP (16.25), or weak formulation, is given by

$$a(y,\phi) = (f,\phi) \quad \forall \phi \in C_0^1(\bar{I}_0), \tag{16.26}$$

where

$$a(y,\phi) = \int_0^1 [a(x)y'\phi' + q(x)y(x)\phi(x)]dx, \tag{16.27}$$

$$(f,\phi) = \int_0^1 f(x)\phi(x)dx. \tag{16.28}$$

Divide an interval $[0,1]$ into M subintervals $K_j = [x_{j-1}, x_j]$ $j = 1,2,\ldots,M$. The corresponding partition $\mathbf{X_M} = [x_0, x_1, \ldots, x_{M-1}, x_M]$ $(M \geq 2)$. To implement the numerical method for solving BVP (16.25), written in the weak form as integral equation (16.26), we replace functions $a(x)$, $y(x)$, $q(x)$ and $f(x)$, for $x \in \bar{I}_0 = [0,1]$, with their projections in the $(M-1)$-dimensional space $S_h = S_h(\mathbf{X_M})$ by piecewise linear functions with respect to a given partition $\mathbf{X_M} = [x_0, x_1, \ldots, x_{M-1}, x_M]$ $(M \geq 2)$ and (16.26) with a finite-dimensional approximation based on piecewise linear finite element (16.6).

16.2.2 Finite-dimensional approximation

Formulate a finite-dimensional problem which approximates BVP (16.25) or (16.26): find $u_h \in S_h(\mathbf{x_m})$ such that

$$a(u_h, \phi_h) = (f, \phi_h) \quad \forall \phi \in S_h(\mathbf{X_M}). \tag{16.29}$$

Here u_h is given by

$$u_h = \sum_{j=1}^{M-1} U_j \Phi_j(x); \tag{16.30}$$

it can be considered as the projection (16.23)

$$P_M(u) = \sum_{i=1}^{M-1} u(x_i)\Phi_i(x) \tag{16.31}$$

of the unknown solution $u(x) \in C_0^1(\bar{I}_0)$ of BVP (16.25) in $(M-1)$-dimensional space $S_h = S_h(\mathbf{X_M})$ of piecewise linear functions with respect to partition (16.3)

$$\mathbf{X_M} = [x_0, x_1, \ldots, x_{M-1}, x_M] \quad (M \geq 2).$$

Insert (16.30) into (16.29) to find that (16.29) is equivalent to

$$\sum_{j=1}^{M-1} U_i a(\Phi_j(x), \Phi_i(x)) = (f, \Phi_i), \quad i = 1,2,\ldots,M-1. \tag{16.32}$$

or in the matrix form

$$A\mathbf{U_M} = \mathbf{f}, \tag{16.33}$$

where vector $\mathbf{f} = [f_1, f_2, \ldots, f_{M-1}]$ the load vector,

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1,M-1} \\ a_{21} & a_{22} & \ldots & a_{2,M-1} \\ . & . & \ldots & . \\ a_{M-1,1} & a_{M-1,2} & \ldots & a_{M-1,M-1} \end{bmatrix},$$

or

$$A = \begin{bmatrix} a(\Phi_1,\Phi_1) & a(\Phi_1,\Phi_2) & a(\Phi_1,\Phi_3) & \dots & a(\Phi_1,\Phi_{M-1}) \\ a(\Phi_2,\Phi_1) & a(\Phi_2,\Phi_2) & a(\Phi_2,\Phi_3) & \dots & a(\Phi_2,\Phi_{M-1}) \\ a(\Phi_3,\Phi_1) & a(\Phi_3,\Phi_2) & a(\Phi_3,\Phi_3) & \dots & a(\Phi_3,\Phi_{M-1}) \\ . & . & . & \dots & . \\ a(\Phi_{M-1},\Phi_1) & a(\Phi_{M-1},\Phi_2) & a(\Phi_{M-1},\Phi_3) & \dots & a(\Phi_{M-1},\Phi_{M-1}) \end{bmatrix} \tag{16.34}$$

is a *stiffness matrix*. The size (dimension) of matrix A is equal to $(M-1)\times(M-1)$ and it is a symmetric matrix: $a_{ij} = a_{ji}$.

Note that function $\Phi_j(x)$ vanishes at the endpoints of the interval. The elements of the stiffness matrix A of the BVP is determined by

$$a(\Phi_j(x),\Phi_i(x))) = a(\Phi_i(x),\Phi_j(x))) = \int_0^1 [\Phi_i'\Phi_j' + q(x)\Phi_i(x)\Phi_j(x)\phi(x)]dx \tag{16.35}$$

Some expressions $\Phi_i\Phi_j$ and $\Phi_i'\Phi_j'$ vanish, e.g.

$$(\Phi_j(x),\Phi_i(x)) = (\Phi_i(x),\Phi_j(x)) = \int_0^1 \Phi_j(x)\Phi_i(x)dx = 0, \quad |i-j| \ge 2 \; (i,j = 1,2,\dots,M-1) \tag{16.36}$$

Stiffness matrix A (16.34) of BVP (16.25) is a symmetric diagonal matrix of size $(M-1)\times(M-1)$

$$\mathscr{A} = \begin{bmatrix} c_0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ a & -b & a & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & a & -b & a & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a & -b & a & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & a & -b & a \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & c_N \end{bmatrix} \tag{16.37}$$

with elements

$$a_{ij} = 0, \quad |i-j| \ge 2 \quad (i,j = 1,2,\dots,M-1); \tag{16.38}$$

namely

$$A = \begin{bmatrix} a_{11} & a_{12} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & a_{M-2,M-3} & a_{M-2,M-2} & a_{M-2,M-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & a_{M-1,M-2} & a_{M-1,M-1} \end{bmatrix} \tag{16.39}$$

If $q = const$, we get

$$\begin{aligned} a(\Phi_j(x),\Phi_i(x)) &= a(\Phi_i(x),\Phi_j(x)) = \int_0^1 [\Phi_i'\Phi_j' + q\Phi_i(x)\Phi_j(x)\phi(x)]dx = \\ &= (\Phi_i',\Phi_j') + q(\Phi_i,\Phi_j) \end{aligned} \tag{16.40}$$

and we can rewrite stiffness matrix A (16.34) as a matrix sum

$$A = Q_1 + Q_0, \quad Q_1 = \left[(\Phi_i',\Phi_j')\right], \quad Q_0 = q\left[(\Phi_i,\Phi_j)\right]. \tag{16.41}$$

16.2.3 Uniform partition

Consider an important case of the uniform partition (16.4) when points $x_j = jh$, $j = 0,1\dots,M$, are distributed uniformly with the step $h = \dfrac{1}{M}$ and piecewise linear base elements are determined as

$$\Phi_j(x) = \begin{cases} 0 & x_0 \le x \le x_{j-1}, \\ \frac{x-x_{j-1}}{h} & x_{j-1} \le x \le x_j, \\ \frac{x_{j+1}-x}{h} & x_j \le x \le x_{j+1}, \\ 0 & x_{j+1} \le x \le x_M, \end{cases} \quad j = 1,2\dots M-1. \tag{16.42}$$

The expressions $(\Phi_j(x), \Phi_i(x)) = 0$, $|i-j| \ge 2$ vanish according to (16.36). Nonzero elements are

$$(\Phi'_j, \Phi'_j) = \int_{x_{j-1}}^{x_j} \frac{1}{h^2}dx + \int_{x_j}^{x_{j+1}} \frac{1}{h^2}dx = \frac{2}{h}, \tag{16.43}$$

$$(\Phi'_{j-1}, \Phi'_j) = \int_{x_{j-1}}^{x_j} \frac{1}{h}\left(-\frac{1}{h}\right) dx = -\frac{1}{h}, \tag{16.44}$$

$$(\Phi_j, \Phi_j) = \int_{x_{j-1}}^{x_j} \frac{(x-x_{j-1})^2}{h^2}dx + \int_{x_j}^{x_{j+1}} \frac{(x_{j+1}-x)^2}{h^2}dx = \frac{2h}{3}, \tag{16.45}$$

$$(\Phi_{j-1}, \Phi_j) = \int_{x_{j-1}}^{x_j} \frac{(x-x_{j-1})}{h}\frac{(x_j-x)}{h}dx = \frac{h}{6}, \tag{16.46}$$

$(i,j = 1,2,\dots,M-1)$.

16.2.4 Constant coefficients

If $q = const$, then the stiffness matrix A is rewritten as a sum of tridiagonal matrices

$$Q_1 = \left[(\Phi'_i, \Phi'_j)\right] = \frac{1}{h}\begin{bmatrix} 2 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -1 & 2 \end{bmatrix} \tag{16.47}$$

and

$$Q_0 = [(\Phi_i, \Phi_j)] = q\frac{h}{6}\begin{bmatrix} 4 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 4 \end{bmatrix} \tag{16.48}$$

If $q = 0$, then the stiffness matrix A coincides with matrix Q_1. The finite-dimensional problem (16.25) approximates the following BVP

$$\begin{cases} -y'' = f(x), & x \in I_0 = (0,1), \\ y(0) = 0, & y(1) = 0, \end{cases} \tag{16.49}$$

or

$$A\mathbf{U_M} = \mathbf{f},$$

where

$$A = Q_1 = \left[(\Phi_i', \Phi_j')\right] = \frac{1}{h}\begin{bmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & \cdots & -1 & 2 \end{bmatrix} \tag{16.50}$$

which coincides with the system obtained for BVP (16.49)

$$\begin{aligned} y'' - q(x)y = f(x),\ d_1 < x < d_2, \\ y(d_1) = f_0,\ y(d_2) = f_N. \end{aligned} \tag{16.51}$$

The forward and backward differences are determined as

$$\Delta y_i = y_{x,i} = \frac{y_{i+1} - y_i}{h},\ \nabla y_i = y_{\bar{x},i} = \frac{y_i - y_{i-1}}{h} \tag{16.52}$$

and

$$y'' \approx y_{\bar{x}x} = y_{\bar{x}x,i} = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}, \tag{16.53}$$

$$-y'' \approx -y_{\bar{x}x} = -y_{\bar{x}x,i} = \frac{-y_{i+1} + 2y_i - y_{i-1}}{h^2},$$

and

$$-y_{\bar{x}x,i} = f_i,\ i = 1, 2, \ldots, N-1, \qquad y_0 = 0,\ y_N = 0,$$

or

$$\begin{cases} y_0 &= 0 \\ \frac{1}{h}(-y_{i-1} + 2y_i - y_{i+1}) &= hf_i, \quad 1 \le i \le N-1, \\ y_N &= 0. \end{cases} \tag{16.54}$$

(16.54) is a system of linear equations.

16.2.5 Approximate solution

Let $\mathbf{U} = [U_1, \ldots, U_{M-1}]$ denote the solution of the linear system (16.33). The approximate solution of BVP (16.25) is defined as a solution of corresponding finite-dimensional problem (16.25)

$$u_h(x) = \sum_{j=1}^{M-1} U_j \Phi_j(x) \tag{16.55}$$

The approximate solution u_h is an element of $(M-1)$-dimensional space $S_h = S_h(\mathbf{X_M})$ of piecewise linear functions with respect to partition (16.3) $\mathbf{X_M} = [x_0, x_1, \ldots, x_{M-1}, x_M]$ $(M \ge 2)$.

16.2.6 Error estimate

The error $r = r(h)$ of the approximate solution of BVP (16.25) can be defined as

$$r(h) = ||u_h - y||_2 = \sqrt{\int_0^1 [u_h(x) - y(x)]^2}, \tag{16.56}$$

where $y(x)$ represents the exact solution of BVP (16.25) and h is the maximum length between adjacent nodes.

The error can be calculated approximately as the Euclidean norm

$$r(h) \approx ||\mathbf{U_M} - \mathbf{Y_M}||_2 = \sqrt{\sum_{j=1}^{M-1} (U_j - y_j)^2}, \tag{16.57}$$

i.e., length of the discrepancy vector $\mathbf{U_M} - \mathbf{Y_M}$, where $\mathbf{Y_M} = [y_1, \dots, y_{M-1}]$ with $y_j = y(x_j)$, $j = 1, 2, \dots M-1$, is the projection (16.31) $P_M(y) = \sum_{i=1}^{M-1} y(x_i)\Phi_i(x)$ of the sought for solution $y(x) \in C_0^1(\bar{I}_0)$.

One can also determine the error approximately with the help of the maximum norm

$$r(h) \approx ||\mathbf{U_M} - \mathbf{Y_M}||_c = \max_{1 \le j \le M-1} |U_j - y_j|. \tag{16.58}$$

One can show that the following estimates hold to the relative error

$$\frac{||u_h - y||_2}{||y||_2} \le Ch^2 \tag{16.59}$$

with some constant C. This means that one can solve approximately the BVP using the finite element method with sufficiently small step h.

Example 25 $M = 3$ corresponds to three subintervals $K_1 = [x_0, x_1]$, $K_2 = [x_1, x_2]$ and $K_3 = [x_2, x_3]$; the corresponding partition is

$$\mathbf{X_3} = [x_0, x_1, x_2, x_3] = [0, x_1, x_2, 1]. \tag{16.60}$$

For this partition, we can define two piecewise linear 'triangular' functions $\Phi_j(x)$ with $j = 1, 2$

$$\Phi_1(x) = \begin{cases} \frac{x}{h_1} & \text{om } 0 = x_0 \le x \le x_1, \\ \frac{x_2 - x}{h_2} = \frac{h_1 + h_2 - x}{h_2} & \text{om } x_1 \le x \le x_2, \end{cases} \quad h_1 = x_1 - x_0 = x_1,\ h_2 = x_2 - x_1; \tag{16.61}$$

$$\Phi_2(x) = \begin{cases} \frac{x - x_1}{h_2} = \frac{x - h_1}{h_2} & \text{om } x_1 \le x \le x_2, \\ \frac{x_3 - x}{h_3} = \frac{1 - x}{h_3} & \text{om } x_1 \le x \le x_2 = 1, \end{cases} \quad h_3 = x_3 - x_2 = 1 - x_2,\ h_2 = x_2 - x_1. \tag{16.62}$$

which have values (as (16.7))

$$\Phi_1(x_1) = 1, \quad \Phi_1(x_0) = \Phi_1(x_2) = 0, \quad \Phi_2(x_2) = 1, \quad \Phi_2(x_1) = \Phi_2(x_3) = 0 \tag{16.63}$$

and are not equal to 0 in the intervals $[x_0, x_2] = K_1 \cup K_2$ and $[x_1, x_3] = K_2 \cup K_3$.

Let $h_1 = h_2 = h_3 = h = 1/3$ and $q = const$. The stiffness matrix is

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

or

$$A = \left[(\Phi_i', \Phi_j') + q(\Phi_i, \Phi_j)\right] = Q_1 + Q_0 = \frac{1}{h}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} + q\frac{h}{6}\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} \frac{2}{h} + \frac{2qh}{3} & \frac{qh}{6} - \frac{1}{h} \\ \frac{qh}{6} - \frac{1}{h} & \frac{2}{h} + \frac{2qh}{3} \end{bmatrix}.$$

Example 26 In the case of $M = 4$ we obtain a system of linear equations with $M-1=3$ unknowns U_1, U_2, U_3 and 3 equations

$$\begin{aligned} a_{11}U_1 + a_{12}U_2 + 0\times U_3 &= f_1, \\ a_{21}U_1 + a_{22}U_2 + a_{23}U_3 &= f_2 \\ 0\times U_1 + a_{32}U_2 + a_{33}U_3 &= f_3 \end{aligned} \tag{16.64}$$

Rewriting system (16.64) in the matrix form, we obtain

$$A\mathbf{U} = \mathbf{f} \tag{16.65}$$

with tridiagonal matrix A (16.39) of size 3×3

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{bmatrix} \tag{16.66}$$

Example 27 Solve the BVP

$$\begin{cases} -y'' + 4y = 2, & 0 < x < 1, \\ y(0) = 0, & y(1) = 0 \end{cases} \tag{16.67}$$

with the help of the finite element method (uniform partition) by reducing it to a system of linear equations with three unknowns. Calculate the approximate solution u_h and determine the (approximate) error $||u_h - y||$ where $y(x)$ is an exact solution of (16.67).

Solution. The weak formulation of BVP(16.67) is

$$a(y,\phi) = (f,\phi) \quad \forall \phi \in C_0^1(\bar{I}_0), \tag{16.68}$$

where

$$a(y,\phi) = \int_0^1 [y'\phi' + 4y(x)\phi(x)]dx = \int_0^1 y'\phi' dx + 4\int_0^1 y(x)\phi(x)dx = (y',\phi') + 4(y,\phi), \tag{16.69}$$

$$(f,\phi) = 2\int_0^1 \phi(x)dx. \tag{16.70}$$

The finite-dimensional problem, which approximates BVP (16.67) or equal weak problem (16.68), is reduced to a system of linear equations with $M-1=3$ unknowns U_1, U_2, U_3 and three equations. In the case $M=4$ we obtain four subintervals

$$\begin{aligned} K_1 &= [x_0,x_1] = [0,h], \quad K_2 = [x_1,x_2] = [h,2h], \\ K_3 &= [x_2,x_3] = [2h,3h], \quad K_4 = [x_3,x_4] = [3h,4h] = [3h,1] \end{aligned} \tag{16.71}$$

with uniform partition

$$\mathbf{X_4} = [x_0,x_1,x_2,x_3,x_4] = [0,x_1,x_2,x_3,1] = [0,h,2h,3h,4h] = h[0,1,2,3,4], \quad h = 0.25. \tag{16.72}$$

For this partition, we can define a piecewise linear base element $\Phi_j(x)$ according to (16.6) with $j = 1,2,3$. The linear system of equation $A\mathbf{U} = \mathbf{f}$ with three unknowns approximates BVP (16.67).

The tridiagonal stiffness matrix A has a size 3×3. We have $q = const = 4$; the stiffness matrix A is a sum of symmetric tridiagonal matrices

$$Q_1 = \left[(\Phi_i', \Phi_j')\right]_{i,j=1}^{3} = \frac{1}{h}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 8 & -4 & 0 \\ -4 & 8 & -4 \\ 0 & -4 & 8 \end{bmatrix}, \tag{16.73}$$

$$Q_0 = [(\Phi_i, \Phi_j)] = 4\frac{h}{6}\begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 2/3 & 1/6 & 0 \\ 1/6 & 2/3 & 1/6 \\ 0 & 1/6 & 2/3 \end{bmatrix}, \tag{16.74}$$

and

$$\begin{aligned} A &= \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{bmatrix} = \left[(\Phi_i', \Phi_j') + 4(\Phi_i, \Phi_j)\right] = Q_1 + Q_0 = \\ &= \begin{bmatrix} 8 & -4 & 0 \\ -4 & 8 & -4 \\ 0 & -4 & 8 \end{bmatrix} + \begin{bmatrix} 2/3 & 1/6 & 0 \\ 1/6 & 2/3 & 1/6 \\ 0 & 1/6 & 2/3 \end{bmatrix} = \begin{bmatrix} 26/3 & -23/6 & 0 \\ -23/6 & 26/3 & -23/6 \\ 0 & -23/6 & 26/3 \end{bmatrix} = \\ &= \frac{1}{6}\begin{bmatrix} 52 & -23 & 0 \\ -23 & 52 & -23 \\ 0 & -23 & 52 \end{bmatrix}. \end{aligned}$$

The right side of system is determined as (16.70)

$$f_i = (f, \Phi_i) = 2\int_0^1 \Phi_i(x)dx = 2\int_0^1 \Phi_i(x)dx = 2\int_{x_{i-1}}^{x_{i+1}} \Phi_i(x)dx = 2h = 0.5, \quad i = 1,2,3.$$

Now we can write the linear system (16.64) $A\mathbf{U} = \mathbf{f}$ with three unknowns which approximates (16.67)

$$\begin{aligned} 52U_1 - 23U_2 &= 3, \\ -23U_1 + 52U_2 - 23U_3 &= 3 \\ -23U_2 + 52U_3 &= 3 \end{aligned} \tag{16.75}$$

(we multiply both sides by 6).

Solve this system using the Gaussian elimination:

$$\begin{aligned} 23U_1 - (23^2/52)U_2 &= 3(23/52), \\ -23U_1 + 52U_2 - 23U_3 &= 3 \\ -23U_2 + 52U_3 &= 3 \end{aligned}$$

$$\begin{aligned} 23U_1 - (23^2/52)U_2 &= 3(23/52), \\ (52 - (23^2/52))U_2 - 23U_3 &= 3(1 + (23/52)) \\ -23U_2 + 52U_3 &= 3 \end{aligned}$$

$$\begin{aligned} 23U_1 - (23^2/52)U_2 &= 3(23/52), \\ 23U_2 - 23^2/(52 - (23^2/52))U_3 &= 3 \cdot 23(1 + (23/52))/(52 - (23^2/52)) \\ -23U_2 + 52U_3 &= 3 \end{aligned}$$

$$
\begin{aligned}
23U_1 - (23^2/52)U_2 &= 3(23/52), \\
23U_2 - 23^2/(52-(23^2/52))U_3 &= 3\cdot 23(1+(23/52))/(52-(23^2/52)) \\
(52-23^2/(52-(23^2/52)))U_3 &= 3+3\cdot 23(1+(23/52))/(52-(23^2/52))
\end{aligned}
$$

The solution of system (16.75) is

$$
\begin{aligned}
U_1 &= \frac{225}{1646} = 0.137 \\
U_2 &= \frac{294}{1646} = 0.179 \\
U_3 &= \frac{225}{1646} = 0.137
\end{aligned}
$$

The error is approximately calculated using the Euclidean norm (16.57)

$$
r(h) \approx ||\mathbf{U_M} - \mathbf{Y_M}||_2 = \sqrt{\sum_{j=1}^{3}(U_j - y_j)^2}. \tag{16.76}
$$

The exact solution of (16.67) is

$$
\begin{aligned}
y(x) &= Ae^{2x} + Be^{-2x} + \frac{1}{2}, && (16.77) \\
A &= \frac{1}{2}\frac{1-e^{-2}}{e^{-2}-e^{2}} = -0.060, \\
B &= \frac{1}{2}\frac{e^{2}-1}{e^{-2}-e^{2}} = -0.440. && (16.78)
\end{aligned}
$$

Projection (16.23) $P_M(f) = \sum_{i=1}^{3} y(x_i)\Phi_i(x)$ can be identified with 3-dimensional vector (16.24)

$$
\begin{aligned}
\mathbf{Y_M} &= [y_1, y_2, y_3], \\
y_i &= y(x_i) = y(ih) = y\left(\frac{i}{4}\right) = Ae^{i/2} + Be^{-i/2} + \frac{1}{2},\ i = 1,2,3.
\end{aligned}
$$

We have

$$
\begin{aligned}
y_1 &= 0.5 - 0.06e^{1/2} - 0.44e^{-1/2} = 0.133, \\
y_2 &= 0.5 - 0.06e - 0.44e^{-1} = 0.175, \\
y_1 &= 0.5 - 0.06e^{3/2} - 0.44e^{-3/2} = 0.133.
\end{aligned}
$$

The target error

$$
\begin{aligned}
r(h) &\approx ||\mathbf{U_M} - \mathbf{Y_M}||_2 = \sqrt{\sum_{j=1}^{3}(U_j - y_j)^2} = \\
&= \sqrt{(0.137-0.133)^2 + (0.179-0.175)^2 + (0.137-0.133)^2} = && (16.79) \\
&= \sqrt{3\cdot 0.004^2} = \sqrt{12\cdot 10^{-6}} = 3.464\cdot 10^{-3} \approx 0.003.
\end{aligned}
$$

The error can also be determined approximately with the help of the maximum norm (16.58)

$$
r(h) \approx ||\mathbf{U_M} - \mathbf{Y_M}||_c = \max_{1\le j\le 3}|U_j - y_j| = 0.004. \tag{16.80}
$$

16.3 Problems

16.3.1 Problem

Define uniform partition of the interval $[0,1]$ and corresponding piecewise linear basic elements Φ_j.

16.3.2 Problem

Draw the piecewise linear basic elements Φ_j in the cases of a uniform partition of the interval $[0,1]$ on

(a) two subintervals,
(b) three subintervals,
(c) four subintervals.

16.3.3 Problem

Let Φ_j, $i = 1,2,\dots,N$ denote a piecewise linear basic element which corresponds to a uniform partition $\{x_i\}$ on the interval $[0,1]$. Plot the functions

(a) $\Phi_1 + 2\Phi_2 + 3\Phi_3$,
(b) $\Phi_1 - \Phi_2 + \Phi_3 + \Phi_4$,
(c) $y_1\Phi_1 + y_2\Phi_2 + y_3\Phi_3 + y_4\Phi_4$, $y_i = y(x_i)$, $y(x) = x + \dfrac{1}{x+1}$.

16.3.4 Problem

Define uniform partition $\mathbf{X_M} = [x_0, x_1, \dots, x_{M-1}, x_M]$ $(M \geq 2)$ on an interval $[0,1]$ and corresponding piecewise linear basic element Φ and determine the projection (16.23) in $(M-1)$-dimensional space $S_h = S_h(\mathbf{X_M})$ of piecewise linear functions with respect to self-defined partition (se (16.3)) of the functions

(a) $y(x) = \dfrac{1}{x+1}$,
(b) $y(x) = x^2 + e^x$,
(c) $y(x) = \sin x + \ln(x+1)$.

16.3.5 Problem

Using nodes $x_i = i$, $i = 0,1,2,3$, construct a piecewise linear function $F(M;x)$ which coincides with $M+1$ given values $y_i = i$, $i = 0,1,2,3$.

Problem 10.6

Using nodes $x_i = i$, $i = 0,1,2,3,4$, construct a piecewise linear function $F(M;x)$ which coincides with $M+1$ given values $y_i = i^2$, $i = 0,1,2,3,4$.

16.3.6 Problem

Solve the BVP

$$\begin{cases} y'' - y = -x, & 0 < x < 1, \\ y(0) = 0, \quad y(1) = 0 \end{cases} \tag{16.81}$$

with the help of the finite element method in the case of four sub-intervals. Calculate approximate solution u_h and determine (approximately) the error $||u_h - y||$, where $y(x)$ represents the exact solution. Determine the approximate error by calculating the norm of deviation vector.

16.3.7 Problem

Solve the BVP

$$\begin{cases} y'' - 4y = -1, & 0 < x < 1, \\ y(0) = 0, \quad y(1) = 0 \end{cases} \tag{16.82}$$

with the help of the finite element method in the case of four sub-intervals. Calculate approximate solution u_h and determine (approximately) the error $u_h||y||$, where $y(x)$ represents the exact solution.

16.4 Solution of two-dimensional BVPs using the finite element method

16.4.1 Strong formulation

As an example, consider BVP (14.19) for the Poisson equation $-\Delta u = f$ in the unit square

$$\begin{cases} -\Delta u = f(x), \quad u = u(x),\, x = (x_1, x_2), \quad 0 < x_1 < 1, \quad 0 < x_2 < 1, \\ u(0, x_2) = 0, \qquad u(1, x_2) = 0, \\ u(x_1, 0) = 0, \qquad u(x_1, 1) = 0, \end{cases} \tag{16.83}$$

which we can rewrite in the form

$$\begin{cases} \mathscr{A}u = -\Delta u = f(x), \qquad x \in \Omega = \Pi_{1,1} = (0,1) \times (0,1), \\ u|_\Gamma = 0, \end{cases} \tag{16.84}$$

where Γ represents the square boundary curve and $f(x)$ is a given continuous function in the square $\bar{\Omega} = [0,1] \times [0,1]$. Compare (16.84) with (16.25) and then the BVP (14.1) for two-dimensional Laplace equation $\Delta u = 0$ in a square $\Pi_{a,b} = \Pi = \{(x_1, x_2) : 0 < x_1 < a, 0 < x_2 < b\}$ ($\bar{\Pi} = \{x, y) : 0 \le x \le a, 0 \le y \le b\}$)

$$\begin{cases} \Delta u = 0, \qquad u = u(x_1, x_2), \quad 0 < x_1 < a, \quad 0 < x_2 < b, \\ u \in C^2(\Pi) \cap C(\bar{\Pi}), \\ u(0, x_2) = f_1(x_2), \qquad u(a, x_2) = f_2(x_2),\, 0 \le x_2 \le b, \\ u(x_1, 0) = f_3(x_1), \qquad u(x_1, b) = f_4(x_1),\, 0 \le x_1 \le a, \\ f_{3,4}(x_1) \in C([0,a]), \quad f_{1,2}(x_2) \in C([0,b]), \end{cases}$$

which we can rewrite in the form

$$\begin{cases} \mathscr{A}u = -\Delta u = 0, \qquad x \in \Pi_{a,b} = (0,a) \times (0,b), \\ u|_\Gamma = f(x). \end{cases}$$

16.4.2 Weak formulation

Denote by $H_0^1 = H_0^1(\Omega)$ the set of functions $v = v(x_1, x_2)$ whose first partial derivatives is square integrable over Ω and satisfies homogeneous boundary conditions

$$v|_\Gamma = 0. \tag{16.85}$$

Multiplying both sides of the Poisson equation $-\Delta u = f$ by an arbitrary function $v \in H_0^1(\Omega)$ and integrating over the unit square Ω with help of (16.85), we get

$$-\int_\Omega\int v\Delta u dx = \int_\Omega\int \nabla u \cdot \nabla v dx. \tag{16.86}$$

It gives us the variation formulation (weak formulation) (16.84) of BVP (16.25)

$$a(u,v) = (f,v) \quad \forall v \in H_0^1(\bar{\Omega}), \tag{16.87}$$

where

$$a(u,v) = \int_\Omega\int \nabla u \cdot \nabla v dx, \tag{16.88}$$

$$(f,v) = \int_\Omega\int f(x)v(x)dx. \tag{16.89}$$

16.4.3 Triangulation

Divide the unit square Ω into a set of triangles K forming a set $\mathscr{T}_h = \{K\}$

$$\Omega = \cup_{K\in\mathscr{T}_h} K, \tag{16.90}$$
$$h_K = \text{diam}(K), \quad h = \max_{K\in\mathscr{T}_h} h_K. \tag{16.91}$$

Triangles vertices P (nodes) form a set of points such that triangles K form a uniform and homogeneous *triangulating* areas of the Ω.

Define a finite-dimensional space $\mathscr{S}_h = S_h(\mathscr{T}_h)$ consisting of piecewise linear functions

$$\mathscr{S}_h = \{v \in \mathscr{S}_h : \ v \text{ is a line in each triangle } K,\ v|_\Gamma = 0\}. \tag{16.92}$$

Let $\{P_i\}_{i=1}^{M_h} \in \Omega$ be the set of internal nodes that do not belong to the curve Γ.

Define piecewise linear 'pyramid' functions $\Phi_j(x)$ which meet the same conditions as piecewise linear 'triangular' functions (16.7)

$$\Phi_i(P_j) = \begin{cases} 1 & \text{om } i = j, \\ 0 & \text{om } i \neq j, \end{cases} \quad i,j = 1,2\ldots M_h, \tag{16.93}$$

and are different from 0 in every part triangle K having at least one vertex P_i in the Ω.

The set $\{\Phi_j(x)\}$ of piecewise linear elements is a basis in a space $\mathscr{S}_h$, i.e., each piecewise linear function $v_h \in \mathscr{S}_h$ can be written as a linear combination of piecewise linear basic element $\Phi_j(x)$

$$\tilde{v}_h(x) = \sum_{i=1}^{M_h} T_i\Phi_i(x). \tag{16.94}$$

(compare with (16.12)).

16.4.4 Finite-dimensional problems

Formulate a finite-dimensional problem which approximates BVP (16.84) or (16.87): Define $u_h \in \mathscr{S}_h$ such that

$$a(u_h, v_h) = (f, v_h) \quad \forall v_h \in \mathscr{S}_h(\mathscr{T}_h). \tag{16.95}$$

Here

$$u_h = \sum_{j=1}^{M_h} U_j\Phi_j(x) \tag{16.96}$$

and can be considered as projection (16.23)

$$P_M(u) = \sum_{i=1}^{M_h} u(P_i)\Phi_i(x) \tag{16.97}$$

of the target (unknown) solution $u(x) \in H_0^1(\Omega)$ of BVP (16.87) in a M_h-dimensional space $\mathscr{S}_h(\mathscr{T}_h)$ of piecewise linear functions with respect to triangulation $\mathscr{T}_h$.

Inserting (16.96) in (16.95), we obtain an equivalent equation

$$\sum_{j=1}^{M_h} U_i a(\Phi_j(x), \Phi_i(x)) = (f, \Phi_i), \quad i = 1,2,\ldots,M_h. \tag{16.98}$$

(16.98) is a system of linear equations with respect on M_h unknowns U_i (vector $\mathbf{U_M} = [U_1, U_2, \dots, U_{M_h}]$) which can be written in the a matrix form

$$A\mathbf{U_M} = \mathbf{f}, \tag{16.99}$$

where vector $\mathbf{f} = [f_1, f_2, \dots, f_{M_h}]$ is a load factor and stiffness matrix A is symmetric.

We see that the finite difference method may be considered as the simplest version of the finite element method of triangulation to solve a BVPs for the Laplace or Poisson equation. The simplest example is the uniform triangulation of the unit square obtained by dividing the x_1 - and x_2 -interval $[0, 1]$ into N subintervals uniformly with the step $h = \frac{1}{N}$.